Andrei B. Klimov and
Sergei M. Chumakov

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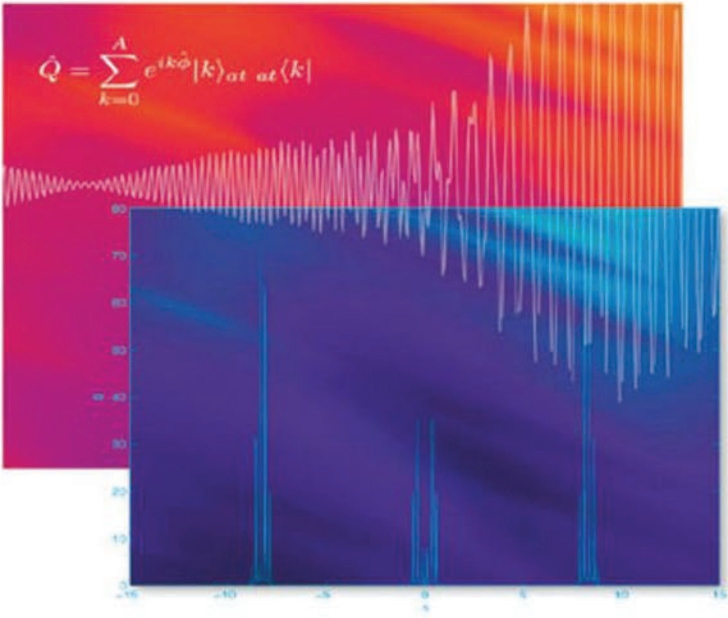
A Group-Theoretical Approach to Quantum Optics

Models of Atom-Field Interactions



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Models of Atom-Field Interactions

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Preface

Quantum Optics is a branch of physics that has developed rapidly over the past few years thanks to the development of experimental techniques that currently allow a single photon to be created and detected, as well as a single atom to be studied in a superconducting cavity. New quantum optics applications such as cold ions, Bose–Einstein condensate, quantum information, and quantum computing motivate the study of the collective properties of quantum systems.

Currently, a number of textbooks on quantum optics are available. They cover ''classical'' quantum optics topics such as quantized field theories and atomic system theories, basic interaction processes between them, and applications such as resonance fluorescence, laser theory, etc. The purpose of this book is not to cover all these topics, which are amply discussed in the standard textbooks. Our goal is different – to show the advantage that may be offered by the algebraic methods applied to problems in quantum optics.

The structure of the book is as follows: Chapters 1–4 are introductory and have been included so that the book is self-consistent and can be read without needing to constantly refer to some other textbooks. (However, these chapters were written in a spirit that emphasizes algebraic methods.) In Chapters 5–10, we describe different models of atom–field interactions, discussing in detail the Jaynes–Cummings and the Dicke models and their generalizations, including the dissipative case using different types of the algebraic perturbation theories. Appendices 1–6 are included to provide the necessary information from the theory of Lie groups and algebras and their representations.

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Atomic Kinematics

In this chapter, we introduce a model for a two-level system, the so-called two-level atom. We consider a collection of many such systems, and introduce different atomic states (Dicke, coherent, and squeezed states). We also discuss a generalization to systems with an arbitrary structure of energy levels.

1.1
Kinematics of an Atom with Two Energy Levels

The state vector for a two-level system can be written as a linear superposition

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \tag{1.1}$$

where vectors $|0\rangle$ and $|1\rangle$ represent respectively the ground and excited states of the system and form a basis in an abstract two-dimensional Hilbert space. The coefficients c_k satisfy the normalization condition $|c_0|^2 + |c_1|^2 = 1$, and $|c_k|^2$ represents the probability of finding the system in the state "k". In a representation where

$$|0\rangle \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad |1\rangle \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

an arbitrary pure state $|\psi\rangle$ is described by a complex two-dimensional vector

$$|\psi\rangle \to \mathbf{c} = \begin{bmatrix} c_1 \\ c_0 \end{bmatrix}$$

Up to a common phase, the amplitudes c_k can be conveniently parameterized as follows:

$$c_1 = \cos\frac{\vartheta}{2}\,\mathrm{e}^{-\mathrm{i}\varphi/2}, \quad c_0 = \sin\frac{\vartheta}{2}\,\mathrm{e}^{\mathrm{i}\varphi/2} \tag{1.2}$$

In the same representation, an arbitrary linear operator $\hat{f}$, acting on the two-level system, has a matrix realization of the form

$$\hat{f} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

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and can be expressed in terms of the Pauli matrices,

$$\sigma_x = \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y = \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}$$
$$\sigma_z = \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad I = \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.3}$$

as

$$\hat{f} = \sum_{j=0}^{3} f_j \sigma_j$$

where the coefficient f_j is given by

$$f_j = \frac{1}{2}\mathrm{Tr}(\hat{f}\sigma_j)$$

If the operator is Hermitian, $\hat{f}^{\dagger} = \hat{f}$, the coefficients f_j are real. Average values of the Pauli matrices in an arbitrary state $|\psi\rangle$ in the parameterization (Equation 1.2) are

$$\langle \sigma_x \rangle = 2\mathrm{Re}\,(c_1 c_0) = \sin\vartheta \cos\varphi$$
$$\langle \sigma_y \rangle = 2\mathrm{Im}(c_1 c_0) = \sin\vartheta \sin\varphi$$
$$\langle \sigma_z \rangle = |c_1|^2 - |c_0|^2 = \cos\vartheta$$

From these, one can infer the average value for $\hat{f}$.

In the two-level atom, the physical interpretation of the operator σ_z can be understood from the Schrödinger equation for a free system:

$$\mathrm{i}\hbar\frac{\mathrm{d}\mathbf{c}}{\mathrm{d}t} = H_0\mathbf{c}, \qquad H_0 = \begin{bmatrix} E_1 & 0 \\ 0 & E_0 \end{bmatrix}$$

If we choose the zero energy to lie exactly between E_0 and E_1, so that $E_0 = -\varepsilon$ and $E_1 = \varepsilon$, the free Hamiltonian of the system is then

$$H_0 = \frac{\omega_0\hbar}{2}\sigma_z, \quad \omega_0 = \frac{E_1 - E_0}{\hbar} = \frac{2\varepsilon}{\hbar}$$

Clearly, the states $|0\rangle$ and $|1\rangle$ are eigenstates of the free Hamiltonian H_0 (these states are sometimes called *bare states*).

The simplest physical realization of a two-level system is clearly a single electron spin in an external magnetic field. To find more applications for a two-level system, let us consider the interaction of a general multilevel atom with an external electromagnetic field. Let us suppose that the atom has only one electron and designate by $|\Psi_n\rangle$, $n = 0, 1, 2, \ldots$ the eigenstates of the free Hamiltonian H_0 of the atom

$$H_0|\Psi_n\rangle = E_n|\Psi_n\rangle \tag{1.4}$$

We assume that the Hamiltonian describing the interaction with the electric field has the simplified form

$$H_{\text{int}} = -\,(\mathbf{d} \cdot \boldsymbol{\mathcal{E}})$$

where $\mathbf{d} = e\mathbf{r}$ is the electric dipole operator, e is the charge, and $\mathbf{r}$ is the electron position vector. In the basis spanned by the $|\Psi_n\rangle$'s, we obtain

$$H_0 = \begin{bmatrix} \cdots & 0 & 0 \\ 0 & E_1 & 0 \\ 0 & 0 & E_0 \end{bmatrix}, \quad H_{\text{int}} = \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & H_{11} & H_{10} \\ \cdots & H_{01} & H_{00} \end{bmatrix}$$

where

$$H_{kn} = \langle \Psi_k | H_{\text{int}} | \Psi_n \rangle = -\left(\boldsymbol{\mathcal{E}} \cdot \int \mathrm{d}\mathbf{r}\, \Psi_k^*(\mathbf{r})\, e\mathbf{r}\, \Psi_n(\mathbf{r}) \right) \equiv -\,(\boldsymbol{\mathcal{E}} \cdot \mathbf{d}_{kn})$$

Since the Hamiltonian is invariant under reflections, the wave functions $|\Psi_n\rangle$ have definite parity. Thus, the diagonal matrix elements of the interaction Hamiltonian vanish: $H_{nn} = -e\boldsymbol{\mathcal{E}} \cdot \langle \Psi_n | \mathbf{r} | \Psi_n \rangle = 0$. The off-diagonal matrix elements H_{kn} describe transitions between the states with energies E_k and E_n, respectively. These transitions have frequency $\omega_{\text{a}} = (E_k - E_n)/\hbar$. To stimulate these transitions, the frequency of the external field must be close to ω_{a}. As the energy levels of an atom are usually not equidistant, the external field does not, in practice, induce transition between levels other than E_k and E_n. This means that, in the matrix for H_{int}, we can focus on a single 2×2 block corresponding to the elements at the intersection of the kth and nth rows with the kth and nth columns. This leads us to the approximation of a two-level atom. We assign an index '0' to the lowest state and an index '1' to the highest state.

The Hamiltonian of the atom takes on the form

$$H = H_0 + H_{\text{int}}, \quad H_0 = \begin{bmatrix} E_1 & 0 \\ 0 & E_0 \end{bmatrix} \tag{1.5}$$

where

$$H_{\text{int}} = -(\boldsymbol{\mathcal{E}} \cdot \mathbf{d}) = \begin{bmatrix} 0 & g \\ g^* & 0 \end{bmatrix}$$

and we have introduced the notation

$$-e\langle 1 | (\boldsymbol{\mathcal{E}} \cdot \mathbf{r}) | 0 \rangle \equiv g = g_{\text{r}} + \mathrm{i}g_{\text{i}}$$

Here, g_{r} and g_{i} are the real and imaginary parts of the complex number g. We obtain

$$\begin{aligned} \langle H_{\text{int}} \rangle &= \left(c_1^* \langle 1| + c_0^* \langle 0| \right) H_{\text{int}} \left(c_1 |1\rangle + c_0 |0\rangle \right) \\ &= g_{\text{r}}(c_1^* c_0 + c_0^* c_1) + \mathrm{i}g_{\text{i}}(c_0 c_1^* - c_0^* c_1) = g_{\text{r}} \langle \sigma_x \rangle - g_{\text{i}} \langle \sigma_y \rangle \end{aligned}$$

Thus, the dipole moment of a two-level atom can be expressed in terms of operators σ_x and σ_y, as follows:

$$\mathbf{d} = \left(\mathrm{Re}\,\boldsymbol{d}\right)\sigma_x - \left(\mathrm{Im}\,\boldsymbol{d}\right)\sigma_y \tag{1.6}$$

where $\boldsymbol{d} = e\langle 1|\mathbf{r}|0\rangle$. The last equation provides a physical interpretation of the operators σ_x and σ_y.

Note that $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{d}$ are vectors in the real physical three-dimensional space. On the other hand, average values of the Pauli matrices $\sigma_{x,y,z}$ form a pseudospin vector in the isotopic space ($\sigma_{x,y,z}$ are sometimes called *energy spin operators* or *isotopic spin operators*). These two spaces should not be confused. (At this point, the difference from spin dynamics in the external magnetic field appears: in the last case the mean values of the spin operators $\boldsymbol{\sigma}$ form a vector in the real space).

In the case of a pure atomic state, the values $\langle\sigma_k\rangle$ are the components of a unit vector in the isotopic space:

$$\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 = 1$$

This vector is called the *Bloch vector*. The tip of this vector is point located on the surface of a unit sphere, called the *Bloch sphere*. The orientation of the Bloch vector on the sphere is uniquely specified by two angles ϑ, φ. To each pure state of a two-level system corresponds a point on the Bloch sphere. The north pole of the sphere corresponds to the excited state $|1\rangle$ and the south pole to the ground state $|0\rangle$. At either pole, the phase φ is not defined.

If the system is in an arbitrary state (pure or mixed), described by a density matrix ρ, where

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{10} \\ \rho_{01} & \rho_{00} \end{bmatrix}$$

the Bloch vector $\mathbf{R} = \mathrm{Tr}(\rho\boldsymbol{\sigma})$, can be written in the following way:

$$R_x = 2\mathrm{Re}\rho_{01}, \quad R_y = 2\mathrm{Im}\rho_{01}, \quad R_z = \rho_{11} - \rho_{00}$$

Inversely, given a Bloch vector, one can recover the density matrix via

$$\rho = \frac{1}{2}\left(1 + \mathbf{R}\boldsymbol{\sigma}\right)$$

It is easy to see that $|\mathbf{R}|^2 = R_x^2 + R_y^2 + R_z^2 = 2\mathrm{Tr}\left(\rho^2\right) - 1 \le 1$, with the equality satisfied only for pure states. This means that, while pure states are represented as points on the surface of the sphere, mixed states are represented as points inside the sphere. The completely mixed atomic state, which is represented by a diagonal matrix $\boldsymbol{\rho} = I/2$, corresponds to the zero-length Bloch vector. In spherical coordinates (r, ϑ, φ), the density matrix elements take the form

$$\rho_{00} = \frac{1}{2}\left(1 - r\cos\vartheta\right), \quad \rho_{01} = \frac{1}{2} r \sin\vartheta\, \mathrm{e}^{\mathrm{i}\varphi}, \quad \rho_{11} = \frac{1}{2}\left(1 + r\cos\vartheta\right)$$

where $r = 1$ corresponds to a pure state.

Note that the basis for linear operators acting in the Hilbert space of the states of a two-level system can be chosen in a different way:

$$|j\rangle\langle k|, \; j, k = 0, 1 \tag{1.7}$$

In this basis, Pauli matrices can be written as follows:

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|, \quad \sigma_y = \mathrm{i}\,(|0\rangle\langle 1| - |1\rangle\langle 0|)\,,$$
$$\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|, \quad I = |1\rangle\langle 1| + |0\rangle\langle 0|$$

1.2
Dicke States

Let us consider a collection of A two-level atoms with the same transition frequency ω_0. We assume that the atoms are distinguishable and do not interact with each other. We designate the nth state of the jth atom as $|j, n\rangle$, $n = 0, 1$. Let k be the number of atoms in the excited state, so that $A - k$ atoms are in the ground state. The possible outcomes of measuring the energy of the system are

$$E = \hbar\omega_0 \left(k - \frac{A}{2}\right), \; k = 0, 1, \ldots, N$$

Thus, the energy levels are equally spaced and the spectrum is bounded. One example of a vector that describes a state with k excited atoms has the form

$$|1, 1\rangle \ldots |k, 1\rangle |k+1, 0\rangle \ldots |A, 0\rangle \tag{1.8}$$

This energy level has a degeneracy of $C_k^A = (A!/k!(A-k)!)$, since any state differing from the state Equation 1.8 only by permutation of the atoms,

$$|j_1, 1\rangle \ldots |j_k, 1\rangle |j_{k+1}, 0\rangle \ldots |j_A, 0\rangle$$

has the same energy. (Here, $A!$ is the total number of atomic permutations and $k!(A-k)!$ is the number of permutations that do not interchange excited and nonexcited atoms.) Now suppose that the states that have the same energy are indistinguishable and consider a normalized symmetric linear combination of all the functions that correspond to the same energy

$$|k, A\rangle = \sqrt{\frac{k!(A-k)!}{A!}} \sum_p |j_1, 1\rangle \ldots |j_k, 1\rangle |j_{k+1}, 0\rangle \ldots |j_A, 0\rangle \tag{1.9}$$

where the sum is made over all the possible permutations that interchange excited and nonexcited atoms. For example, for two atoms these symmetric states are

$$|0, 2\rangle = |1, 0\rangle |2, 0\rangle$$
$$|1, 2\rangle = (|1, 1\rangle |2, 0\rangle + |1, 0\rangle |2, 1\rangle)\,/\sqrt{2}$$
$$|2, 2\rangle = |1, 1\rangle |2, 1\rangle$$

Let us define the *collective operators* for a system of A two-level atoms in the following way:

$$S_{x,y,z} = \frac{1}{2} \sum_{j=1}^{A} \sigma_{x,y,z}^{(j)} \tag{1.10}$$

where $\sigma_{z,y,z}^{(j)}$ are individual operators for each of the atoms. It is obvious that in the collective basis (1.9),

$$S_z |k, A\rangle = \left(k - \frac{A}{2}\right) |k, A\rangle$$

Thus, the operator S_z defines the inversion in the atomic system, that is, the difference between the number of excited and nonexcited atoms. The operators

$$\sigma_{\pm}^{(j)} = \frac{1}{2} \left(\sigma_x^{(j)} \pm \mathrm{i}\sigma_y^{(j)}\right)$$

satisfy the commutation relations

$$[\sigma_+^{(j)}, \sigma_-^{(j)}] = \sigma_z^{(j)}, \qquad [\sigma_z^{(j)}, \sigma_{\pm}^{(j)}] = \pm 2\sigma_{\pm}^{(j)}$$

and act on the basis elements of each atom as

$$\sigma_+^{(j)} |j, 0\rangle = |j, 1\rangle, \quad \sigma_+^{(j)} |j, 1\rangle = 0, \quad \sigma_-^{(j)} |j, 1\rangle = |j, 0\rangle, \quad \sigma_-^{(j)} |j, 0\rangle = 0$$

Thus, the collective operators

$$S_{\pm} = \sum_{j=1}^{A} \sigma_{\pm}^{(j)} = S_x \pm \mathrm{i} S_y \tag{1.11}$$

act on symmetric states (Equation 1.9) as follows:

$$\begin{aligned} S_- |k, A\rangle &= \sqrt{\frac{k!(A-k)!}{A!}} \left(A - k + 1\right) \sum_p |j_1, 1\rangle \ldots |j_{k-1}, 1\rangle |j_k, 0\rangle \ldots |j_A, 0\rangle \\ &= \sqrt{k(A - k + 1)}\, |k - 1, A\rangle \end{aligned}$$

Similarly, one obtains the action of the operator S_+. Finally, the action of the collective operators on the basis of the subspace of symmetric states (1.9) is given as

$$\begin{aligned} S_+ |k, A\rangle &= \sqrt{(k+1)(A-k)}\, |k + 1, A\rangle \\ S_- |k, A\rangle &= \sqrt{k(A - k + 1)}\, |k - 1, A\rangle \\ S_z |k, A\rangle &= \left(\frac{k - A}{2}\right) |k, A\rangle \end{aligned} \tag{1.12}$$

The collective operators $S_{x,y,z}$ satisfy the familiar $su(2)$ algebra commutation relations:

$$[S_x, S_y] = \mathrm{i} S_z\,, \quad [S_y, S_z] = \mathrm{i} S_x\,, \quad [S_z, S_x] = \mathrm{i} S_y \tag{1.13}$$

or, in terms of the operators $S_\pm$

$$[S_z, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = 2S_z \tag{1.14}$$

Since $S_z^2|k,A\rangle = \left(\frac{k-A}{2}\right)^2 |k,A\rangle$, and

$$\left(S_x^2 + S_y^2\right)|k,A\rangle = \frac{1}{2}\left(S_+S_- + S_-S_+\right)|k,A\rangle = \left(Ak - k^2 + \frac{A}{2}\right)|k,A\rangle$$

the state $|k,A\rangle$ is the eigenstate of the *Casimir operator*

$$S^2 = S_z^2 + S_x^2 + S_y^2 = S_z^2 + \frac{1}{2}\left(S_+S_- + S_-S_+\right) \tag{1.15}$$

The corresponding eigenvalue does not depend on k:

$$S^2|k,A\rangle = \frac{A}{2}\left(\frac{A}{2}+1\right)|k,A\rangle$$

Moreover, it is easy to prove that $[S_{x,y,z}, S^2] = 0$. This means that the symmetric states $|k,A\rangle$ form a basis of the $(A+1)$-dimensional *irreducible representation* of the $su(2)$ algebra. The vectors $|k,A\rangle$ are called *Dicke states* [13].

1.3
Atomic Coherent States

Let us again consider a system of A (distinguishable) two-level atoms that do not interact with each other. Now we suppose that all the atoms are in the same quantum state, so that the state of the whole system is simply the product of the states of individual atoms. In the polar parameterization (Equation 1.2), we obtain

$$|\vartheta,\varphi\rangle = \prod_{j=1}^{A}\left(\cos\frac{\vartheta}{2}\,\mathrm{e}^{-\mathrm{i}\varphi/2}|j,1\rangle + \sin\frac{\vartheta}{2}\,\mathrm{e}^{\mathrm{i}(\varphi/2)}|j,0\rangle\right) \tag{1.16}$$

It is easy to see that this product transforms into

$$|\vartheta,\varphi\rangle = \sum_{k=0}^{A}\cos^k\frac{\vartheta}{2}\,\sin^{A-k}\frac{\vartheta}{2}\,\mathrm{e}^{\mathrm{i}((A-2k)\varphi/2)}\sum_{p}|j_1,1\rangle\ldots|j_k,1\rangle|j_{k+1},0\rangle\ldots|j_A,0\rangle$$

where, on the right hand side, one can recognize the Dicke state $|k,A\rangle$

$$|\vartheta,\varphi\rangle = \mathrm{e}^{\mathrm{i}\frac{A}{2}\varphi}\sum_{k=0}^{A}\sqrt{\frac{A!}{k!(A-k)!}}\,\mathrm{e}^{-\mathrm{i}k\varphi}\cos^k\frac{\vartheta}{2}\,\sin^{A-k}\frac{\vartheta}{2}\,|k,A\rangle \tag{1.17}$$

$$= \mathrm{e}^{-\mathrm{i}\varphi S_z}\sum_{k=0}^{A}\sqrt{\frac{A!}{k!(A-k)!}}\cos^k\frac{\vartheta}{2}\,\sin^{A-k}\frac{\vartheta}{2}\,|k,A\rangle$$

The state $|\vartheta,\varphi\rangle$ is, in fact, a coherent state for the $su(2)$ group [14] (see Appendix 11.2). Let us replace $\vartheta = \theta + \pi$ in Equation 1.17 and introduce the state [15, 16]

$$|\xi\rangle = \mathrm{e}^{-\mathrm{i}\varphi A/2}|\vartheta = \theta + \pi, \varphi\rangle, \quad \xi = -\tan\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\varphi}$$

The state $|\xi\rangle$ can be generated from vector $|0, A\rangle$, describing a state with no atom excited, by the displacement operator $D(z)$:

$$D(z) = \exp\left(zS_+ - z^* S_-\right), \quad z = -\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi} \tag{1.18}$$

$$D^\dagger(z) = D^{-1}(z) = D(-z)$$

The displacement operator can be rewritten in different ways:

$$\begin{aligned} D(z) &= \exp(-\mathrm{i}\varphi S_z)\exp(-\mathrm{i}\theta S_y)\exp(\mathrm{i}\varphi S_z) \\ &= \exp(\xi S_+)\exp\left[\ln\left(1 + |\xi|^2\right) S_z\right]\exp(-\xi^* S_-) \end{aligned}$$

Thus, the ket $|\xi\rangle$ takes the form

$$|\xi\rangle = D(z)|0, A\rangle = \left(1 + |\xi|^2\right)^{-A/2}\exp\left[\xi\, S_+\right]|0, A\rangle \tag{1.19}$$

It is easy to write the expansion of the coherent state $|\xi\rangle$ in the basis of the Dicke states:

$$|\xi\rangle = \frac{1}{\left(1 + |\xi|^2\right)^{A/2}}\sum_{k=0}^{A}\xi^k\sqrt{\frac{A!}{k!(A-k)!}}|k, A\rangle \tag{1.20}$$

The coherent state (1.17) can also be obtained by applying the operator $\exp(-\mathrm{i}\varphi S_z)$ $\exp(-\mathrm{i}\vartheta S_y)$ to the fully excited state $|A, A\rangle$:

$$|\vartheta, \varphi\rangle = \exp(-\mathrm{i}\varphi S_z)\exp(-\mathrm{i}\vartheta S_y)|A, A\rangle$$

To understand the physical properties of the atomic coherent states, we calculate the expectation values of the operators $S_{x,y,z}$. It follows directly from Equation 1.20 that

$$\langle\xi|S_+|\xi\rangle = \frac{1}{\left(1 + |\xi|^2\right)^A}\sum_{k,n=0}^{A}\frac{\xi^{*k}\xi^n}{k!n!}\langle A, 0|S_-^k S_+ S_+^n|0, A\rangle$$

and, using the relation

$$\langle A, 0|S_-^k S_+^{n+1}|0, A\rangle = \delta_{k,n+1}\frac{k!A!}{(A-k)!}$$

we immediately obtain

$$\langle\xi|S_+|\xi\rangle = \frac{\xi^*}{\left(1 + |\xi|^2\right)^A}\sum_{k=1}^{A}\frac{|\xi|^{2(k-1)}A!}{(k-1)!(A-k)!} = \frac{A\,\xi^*}{1 + |\xi|^2}$$

or, in the angular representation

$$\langle \xi | S_+ | \xi \rangle = -\frac{A}{2} \sin\theta \, e^{i\varphi} = \langle \xi | S_- | \xi \rangle^*$$

If we now observe that

$$S_+ |\xi\rangle = \frac{1}{\xi} \left(S_z + \frac{A}{2} \right) |\xi\rangle, \quad S_- |\xi\rangle = \xi \left(\frac{A}{2} - S_z \right) |\xi\rangle \tag{1.21}$$

we can easily evaluate $\langle \xi | S_z | \xi \rangle = \xi \langle \xi | S_+ | \xi \rangle - A/2$, and finally, we have

$$\langle \xi | S_x | \xi \rangle = -\frac{A}{2} \sin\theta \cos\varphi, \quad \langle \xi | S_y | \xi \rangle = -\frac{A}{2} \sin\theta \sin\varphi, \quad \langle \xi | S_z | \xi \rangle = -\frac{A}{2} \cos\theta$$

or for the coherent states (Equations 1.16 and 1.17),

$$\langle \vartheta, \varphi | \mathbf{S} \, |\vartheta, \varphi\rangle = \frac{A}{2} \mathbf{n}, \quad \mathbf{n} = (\sin\vartheta \cos\varphi, \sin\vartheta \sin\varphi, \cos\theta) \tag{1.22}$$

where $\mathbf{S} = (S_x, S_y, S_z)$ and $\mathbf{n}$ is the unit vector that determines the direction of the classical angular momentum. The averages of the operators S_j then form a vector in the direction of the classical momentum (collective Bloch vector):

$$\mathbf{R} = \langle \vartheta, \varphi | \mathbf{S} \, |\vartheta, \varphi\rangle = \frac{A}{2} \mathbf{n}$$

These average values are identical to those obtained for the single atom case, except for the change in the value of the isotopic spin: $1/2 \to A/2$. This suggests that a coherent state $|\theta, \varphi\rangle$ can be interpreted as a symmetric generalization of the one-atom state. In fact, Equation 1.22 follows from the observation that the spin coherent states are eigenstates of the operator $(\mathbf{S} \cdot \mathbf{n})$, as shown in Appendix 11.2.

Fluctuations of the pseudospin projection S_x in the state $|\vartheta, \varphi\rangle$

$$\langle \Delta S_x^2 \rangle = \left\langle S_x^2 - \langle S_x \rangle^2 \right\rangle, \quad \langle \ldots \rangle \equiv \langle \vartheta, \varphi | \ldots | \vartheta, \varphi \rangle$$

can be easily found from Equation 1.17. However, it is more interesting to use the definition (Equation 1.16) directly. We have

$$\langle S_x^2 \rangle = \frac{1}{4} \left\langle \left(\sum_{j=1}^{A} \sigma_x^{(j)} \right)^2 \right\rangle = \frac{1}{4} \left\langle \sum_{j=1}^{A} \left(\sigma_x^{(j)} \right)^2 + \sum_{i \neq j} \sigma_x^{(i)} \sigma_x^{(j)} \right\rangle$$

In the first term on the far right, $\left(\sigma_x^{(j)}\right)^2 = 1$ while the additional term $\left\langle \sigma_x^{(i)} \sigma_x^{(j)} \right\rangle = \left\langle \sigma_x^{(i)} \right\rangle \left\langle \sigma_x^{(j)} \right\rangle = \sin^2\vartheta \cos^2\varphi$, due to the factorized form of the state (1.16). This leads to

$$\langle S_x^2 \rangle = \frac{A}{4} + \frac{A(A-1)}{4} \sin^2\vartheta \cos^2\varphi$$

Using Equation 1.22, we obtain

$$\langle \Delta S_x^2 \rangle = \frac{A}{4} \left(1 - \sin^2\vartheta \cos^2\varphi \right)$$

Relative fluctuations are

$$\delta_x = \frac{\sqrt{\langle \Delta S_x^2 \rangle}}{\langle S_x \rangle} = \frac{1}{\sqrt{A}} \frac{\sqrt{1 - \sin^2\vartheta \cos^2\varphi}}{\sin\vartheta \cos\varphi}$$

For the states with $\varphi = \pm\,\pi/2$ as well as for the states $|0\rangle$ and $|1\rangle$ corresponding to the values $\vartheta = 0$ and $\vartheta = \pi$ (north and south poles of the Bloch sphere), $\langle S_x \rangle = 0$ and relative fluctuations $\delta_x \to \infty$. For all other coherent states, relative fluctuations δ_x tend toward zero in the limit of a large number of atoms, $A \to \infty$. States with the lowest fluctuations of S_x correspond to the points on the equator of the Bloch sphere, $\vartheta = \pi/2$. In this case, half of the atoms are excited. Fluctuations $\langle \Delta S_x^2 \rangle$ completely disappear when $\vartheta = \pi/2$ and $\varphi = 0$, or $\varphi = \pi$: these two coherent states are eigenstates of the operator S_x. It is worth noting that the directions of the axes of the coordinate system can always be chosen in such a way that fluctuations of a given operator (S_x, S_y or S_z) depend on one angle only. For example, in the case of operator S_x, this angle is between the x axis and the direction of $\mathbf{n}$.

Finally, fluctuations for the operators $S_{x,y,z}$ in the atomic coherent states $|\vartheta, \varphi\rangle$ are

$$(\Delta S_x)^2 = \frac{A}{4}\left(1 - \sin^2\vartheta \cos^2\varphi\right) \tag{1.23}$$

$$(\Delta S_y)^2 = \frac{A}{4}\left(1 - \sin^2\vartheta \sin^2\varphi\right)$$

$$(\Delta S_z)^2 = \frac{A}{4}\left(1 - \cos^2\vartheta\right)$$

One can observe that the square fluctuations in the coherent states are A times larger than in the case of one atom. Further note that

$$(\Delta S_x)^2 + \left(\Delta S_y\right)^2 + (\Delta S_z)^2 = \langle S^2 \rangle - \langle \mathbf{S} \rangle^2 = \frac{A}{2}$$

It is also easy to calculate the fluctuations of the component of the vector operator $\mathbf{S}$ in the direction perpendicular to the direction $\mathbf{n}$ of the average spin $\mathbf{R} = \langle \vartheta, \varphi | \mathbf{S} | \vartheta, \varphi \rangle$. For this goal one can write

$$\mathbf{S} = \mathbf{n}(\mathbf{S} \cdot \mathbf{n}) + \mathbf{S}_\perp$$

where $\mathbf{S}_\perp$ is a two-dimensional vector that belongs to the plane tangent to the sphere at point $(\sin\vartheta \cos\varphi, \sin\vartheta \sin\varphi, \cos\vartheta)$:

$$\mathbf{S}_\perp = \mathbf{S}_\perp^{(1)} + \mathbf{S}_\perp^{(2)}, \quad \mathbf{S}_\perp^{(1)} \perp \mathbf{S}_\perp^{(2)}$$

Using Equation 1.22, we have

$$\langle \vartheta, \varphi | \mathbf{S}_\perp | \vartheta, \varphi \rangle = 0 \tag{1.24}$$

As stated earlier, $|\vartheta, \varphi\rangle$ is the eigenvector of the operator $(\mathbf{S} \cdot \mathbf{n})$:

$$(\mathbf{S} \cdot \mathbf{n}) |\vartheta, \varphi\rangle = \frac{A}{2} |\vartheta, \varphi\rangle$$

thus,

$$\langle(\Delta\mathbf{S}_{\perp})^2\rangle = \langle S^2\rangle - \langle(\mathbf{S}\cdot\mathbf{n})\rangle^2 = \frac{A}{2}\left(\frac{A}{2}+1\right) - \left(\frac{A}{2}\right)^2 = \frac{A}{2}$$

where the reader is reminded that $\langle\ldots\rangle$ is taken over the coherent state $|\vartheta,\varphi\rangle$.

From Equation 1.19, we conclude that, for the coherent states, the fluctuations of the components of vector $\mathbf{S}_{\perp}$ are equal. This is obvious for the coherent state $|0, A\rangle$, i.e. the coherent state located at $\theta = 0$ (or, equivalently, at $\vartheta = \pi$). As physical properties of the coherent states do not depend on rotations, these properties are not changed by the action of the operator $D(\theta,\varphi)$. Thus, we obtain

$$\langle\vartheta,\varphi|\left(\Delta\mathbf{S}_{\perp}^{(1)}\right)^2|\vartheta,\varphi\rangle = \langle\vartheta,\varphi|\left(\Delta\mathbf{S}_{\perp}^{(2)}\right)^2|\vartheta,\varphi\rangle = \frac{A}{4} \qquad (1.25)$$

This equation, taken in conjunction with Equation 1.22, allows us to visualize a coherent atomic state as a symmetric segment of radius $\sqrt{A}/2$ on a Bloch sphere of radius $A/2$, centered at the point determined by the vector $\mathbf{n}$ of Equation 1.22. In the classical limit, where $A \to \infty$, the radius of this segment divided by the Bloch sphere radius is proportional to $1/\sqrt{A} \to 0$. This means that the relative fluctuations in the directions perpendicular to the Bloch vector disappear.

It is important to note that, for a system described by a coherent state, the number of excitations is not well defined, except at the poles, where $\vartheta = 0$ or π. From Equation 1.17, one can obtain the probability of finding k excitations in the system

$$p_k = |\langle\vartheta,\varphi|k, A\rangle|^2 = \frac{A!}{k!(A-k)!}\, x^k\,(1-x)^{A-k}$$

where $x = \sin^2\vartheta/2$. This expression corresponds to the binomial distribution, which tends to the Poisson distribution when $A \to \infty$ (and $A\sin^2\vartheta/2$ is a finite number). Assuming that each excitation has the energy $\hbar\omega_0/2$ and each nonexcited state has the energy $-\hbar\omega_0/2$, we find the average energy in a coherent state:

$$E = \frac{\hbar\omega_0}{2}\sum_k (2k - A)p_k = \hbar\omega_0\,\frac{A}{2}\cos\vartheta$$

This result may be obtained by calculating the mean energy as the average of the free Hamiltonian over the coherent state and using Equation 1.22.

$$E = \langle\vartheta,\varphi|H_0|\vartheta,\varphi\rangle\,, \quad H_0 = \hbar\omega_0 S_z$$

It can easily be seen that at the poles, $\vartheta = 0$ and $\vartheta = \pi$, the energy takes its maximum and minimum values, while for points on the equator, $\vartheta = \pi/2$, the mean energy equals to zero.

The free evolution of the coherent state $|\vartheta,\varphi\rangle$

$$|\Psi(t)\rangle = U(t)|\vartheta,\varphi\rangle, \quad U(t) = \exp\left(-\mathrm{i}t\,\frac{H_0}{\hbar}\right)$$

is obtained from Equation 1.17 replacing $\varphi \to \varphi + \omega_0 t$, since

$$H_0|k, A\rangle = E_k|k, A\rangle, \quad E_k = \hbar\omega_0\left(k - \frac{A}{2}\right)$$

1.4
Squeezed Atomic States

As we have seen, if an atomic system is prepared in a coherent state, the square fluctuations of the components of the vector **S** in any direction perpendicular to $\langle \mathbf{S} \rangle$ are equal to $A/4$. Thus fluctuations are uniformly distributed, that is, fluctuations of any component lying in the plane tangent to the sphere and perpendicular to the average spin vector are the same. Physically, this is clear since in a coherent state, individual atoms are all in the same state (1.16) and, thus, the square of the fluctuation of the component of **S** is simply the sum of the fluctuations corresponding to different atoms. One may say that the atoms are *not correlated,* in the sense that the mean value of any observable can be calculated by averaging over individual atomic states.

In principle, it is possible to form a *correlated* superposition of atomic states in such a way that fluctuations are no longer distributed uniformly in the tangent plane and, in some directions, the fluctuations would be smaller than $A/4$, at the expense of the fluctuations in the other perpendicular directions, which will be larger than $A/4$.

To correlate the atoms with each other, a transformation generated by a nonlinear combination of generators of the $su(2)$ algebra representation is required, since linear transformations only rotate individual isotopic spins and would not produce between the atoms a correlation of the type described above. One way to introduce a correlation into a collective atomic system consists in inserting a nontrivial phase in Equation 1.17, that depends on a set of parameters $\boldsymbol{\mu}$

$$|\vartheta, \varphi, \boldsymbol{\mu}\rangle = \mathrm{e}^{\mathrm{i}A/2\varphi} \sum_{k=0}^{A} \sqrt{\frac{A!}{k!(A-k)!}}\, \mathrm{e}^{-\mathrm{i}k\,\varphi - \mathrm{i}f(k,\boldsymbol{\mu})} \cos^k \frac{\vartheta}{2} \sin^{A-k} \frac{\vartheta}{2}\, |k, A\rangle$$

Here $f(k, \boldsymbol{\mu})$ is a nonlinear function of k, which is equivalent to the application to the coherent state (1.17) of a unitary transformation, which is nonlinear in the collective inversion operator S_z: $|\vartheta, \varphi, \boldsymbol{\mu}\rangle = U_s(\mu)|\vartheta, \varphi\rangle$. A simple model for such a transformation was discussed in [17]

$$U_s(\mu) = \exp\left[-\mathrm{i}\mu S_z^2\right] \tag{1.26}$$

This model is analogous to the field-mode evolution in the presence of the Kerr medium; however, it has a finite number of states and we call it the *finite Kerr medium.* Upon applying the transformation (1.26) to the atomic operators $S_\pm$, nonlinear phases are generated:

$$S_+(\mu) = U_s^\dagger S_+ U_s = S_+ \exp\left[2\mathrm{i}\mu\left(S_z + \frac{1}{2}\right)\right]$$

$$S_-(\mu) = U_s^\dagger S_- U_s = \exp\left[-2\mathrm{i}\mu\left(S_z + \frac{1}{2}\right)\right] S_-$$

which leads to a redistribution of quantum fluctuations in the plane tangent to the Bloch sphere. Note that the operators S_z^2 and $\{S_+, S_-\} \equiv S_+S_- + S_-S_+$ commute with S_z, and are thus unchanged by the transformation of (Equation 1.26). The value of fluctuations in the tangent plane are obviously invariant with respect to rotations about the z axis. In other words, the distribution of fluctuations in the tangent plane does not depend on the phase φ of the Bloch vector ; so, we assume, without loss of generality, that $\varphi = 0$. The direction in the plane tangent to the Bloch sphere at the point with coordinates $(\vartheta,\ \varphi = 0)$ is determined by a unit vector

$$\mathbf{n}_\perp = (-\cos\vartheta\sin\delta,\ \cos\delta,\ \sin\vartheta\sin\delta), \quad \mathbf{n} \perp \mathbf{n}_\perp$$

where the unit vector $\mathbf{n} = (\sin\vartheta, 0, \cos\vartheta)$ defines the direction of the collective Bloch vector (Equation 1.22) and δ is the angle between the vector $\mathbf{n}_\perp$ and the positive direction of the y axis. Fluctuations of the projection of the vector $\mathbf{S}$ in the direction $\mathbf{n}_\perp$,

$$s_\delta = (\mathbf{S}\cdot\mathbf{n}_\perp) = -S_x\cos\vartheta\sin\delta + S_y\cos\delta + S_z\sin\delta\sin\vartheta \tag{1.27}$$

depend on the angle ϑ that determines the position on the Bloch sphere. For a given value of the angle ϑ, we can find the direction of maximum squeezing by changing the angle δ. It is easy to see that

$$\left[s_\delta, s_{\delta+\pi/2}\right] = \mathbf{S}\cdot\mathbf{n}$$

Formally, the atomic squeezing can be quantified by the ratio

$$\zeta = \frac{\min\sigma_{s_\delta}}{\sqrt{A/4}} \tag{1.28}$$

where $\min\sigma^2_{s_\delta} = \min\left[\langle s^2_\delta\rangle - \langle s_\delta\rangle^2\right]$ is the smallest variation of the isotopic spin component normal to the mean spin $\langle\mathbf{S}\rangle$. Clearly, in the coherent state $\zeta = 1$, and the state is squeezed if $\zeta < 1$.

Let us find, for example, the amount of squeezing generated in a set of A atoms by the transformation (1.26) applied to a coherent atomic state $|\pi/2, 0\rangle$, located at the equator of the Bloch sphere (i.e. $\vartheta = \pi/2$):

$$|\psi_c\rangle = U_s(\mu)|\pi/2, 0\rangle \tag{1.29}$$

From Equation 1.27, we obtain

$$\langle\psi_c|s_\delta|\psi_c\rangle = \left\langle S_z\sin\delta + \frac{\cos\delta}{2i}\left(S_+(\mu) - S_-(\mu)\right)\right\rangle_0$$

$$\langle\psi_c|s^2_\delta|\psi_c\rangle = \langle S^2_z\rangle_0\sin^2\delta - \left\langle S^2_+(\mu) + S^2_-(\mu) - \{S_+, S_-\}\right\rangle_0\frac{\cos^2\delta}{4}$$
$$+\langle\{S_z, S_+(\mu)\} - \{S_z, S_-(\mu)\}\rangle_0\frac{\sin 2\delta}{4i}$$

where $\langle\ldots\rangle_0$ means the average over the coherent state $|\pi/2, 0\rangle$. Following Equation 1.17, we have

$$\langle S_z\rangle_0 = 0, \quad \langle S^2_z\rangle_0 = \frac{A}{4}, \quad \langle\{S_+, S_-\}\rangle_0 = \frac{A(A-1)}{2}, \quad \langle\exp[2i\mu S_z]\rangle_0 = \cos^A\mu$$

It is easy to find, using the relations (1.21), that

$$\begin{aligned} \left\langle S_+(\mu)\right\rangle_0 &= \frac{A}{2}\cos^{A-1}\mu \\ \left\langle \left[S_+(\mu)\right]^2\right\rangle_0 &= \frac{A(A-1)}{4}\cos^{A-2}2\mu \\ \left\langle \{S_z, S_+(\mu)\}\right\rangle_0 &= i\,\frac{A(A-1)}{2}\cos^{A-2}\mu\,\sin\mu \end{aligned} \tag{1.30}$$

We obtain the following expression for the fluctuations of the spin projection (1.27) in the state (1.29):

$$\begin{aligned} \left(\Delta s_\delta(\mu)\right)^2 &= \langle\psi_c|s_\delta^2|\psi_c\rangle - \left(\langle\psi_c|s_\delta|\psi_c\rangle\right)^2 \\ &= \frac{A}{4}\left[1+\frac{A-1}{4}\left(a+\sqrt{a^2+b^2}\cos\left(2\delta-2\nu\right)\right)\right] \end{aligned} \tag{1.31}$$

where

$$a = 1-\cos^{A-2}2\mu,\quad b = 4\sin\mu\cos^{A-2}\mu,\quad \tan 2\nu = \frac{b}{a}$$

From Equation 1.31, we can see that the fluctuations are minimized in the direction determined by the angle

$$\delta = \frac{1}{2}\arctan\frac{b}{a}+\frac{\pi}{2} \tag{1.32}$$

and reach the value

$$\left(\Delta s(\mu)\right)^2_{\min} = \frac{A}{4}\left[1-\frac{A-1}{4}\left(\sqrt{a^2+b^2}-a\right)\right] \le \frac{A}{4} \tag{1.33}$$

Obviously, fluctuations in the direction perpendicular to Equation 1.32 are maximized. It is clear that no squeezing can occur in the case of one atom (since this atom would not have another to correlate to). In the particular case of two atoms, the minimum value of the fluctuations (Equation 1.31) may be zero. It may also happen that both the mean spin and $\min\sigma^2_{s_\delta}$ takes zero values at some state (for instance, in the two-atom Dicke state $|1, A=2\rangle$), so that the parameter (Equation 1.28) becomes undefined.

It is instructive to visualize the state (1.29) on the two-dimensional sphere in terms of the so-called Q function (see, e.g. Chapter 10)

$$Q_\mu(\vartheta,\varphi) = \left|\left\langle\vartheta,\varphi\right|\mathrm{e}^{-\mathrm{i}\mu S_z^2}\left|\pi/2,0\right\rangle\right|^2 \tag{1.34}$$

where $|\vartheta,\varphi\rangle$ is the coherent state (1.17), in the limit of large number of atoms, $A\gg 1$. The matrix element appearing in Equation 1.34 takes the form

$$\left\langle\vartheta,\varphi\right|\mathrm{e}^{-\mathrm{i}\mu S_z^2}\left|\pi/2,0\right\rangle = \frac{1}{\left(1+|\xi|^2\right)^{A/2}}\frac{1}{2^{A/2}}\sum_{k=0}^{A}\xi^{*k}C_k^A\,\mathrm{e}^{-\mathrm{i}\mu\left(k-(A/2)\right)^2} \tag{1.35}$$

where $\xi = \cot(\vartheta/2)\,\mathrm{e}^{-\mathrm{i}\varphi}$ and C_k^A are the binomial coefficients. In the limit $A \gg 1$, the binomial coefficients can be approximated by a Gaussian

$$C_k^A \approx \frac{2^{2S}}{\sqrt{\pi S}} \exp\left[-\frac{(k-S)^2}{S}\right] \tag{1.36}$$

where $S = A/2$. Substituting Equation 1.36 in Equation 1.35 and changing the summation index $k - S = n$, we get

$$\langle \vartheta, \varphi | \,\mathrm{e}^{-\mathrm{i}\mu S_z^2} | \pi/2, 0\rangle = \frac{1}{\left(1+|\xi|^2\right)^S} \frac{\xi^S}{2^S} \sum_{n=-S}^{S} \exp\left[-\frac{n^2}{s} - \mathrm{i}\mu n^2 + n \ln|\xi| + in\varphi\right]$$

Using the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x f(x)\, \mathrm{e}^{2\pi i m x}$$

we obtain (extending the summation to infinity),

$$\sum_{n=-S}^{S} \exp\left[-\frac{n^2}{S} - \mathrm{i}\mu n^2 + n \ln|\chi| + in\varphi\right] \approx \sqrt{\frac{\pi}{a}} \sum_{m=-\infty}^{\infty} \mathrm{e}^{(b_m^2/4a)}$$

where

$$a = \frac{1}{S} + \mathrm{i}\mu, \quad b_m = \ln|\xi| + \mathrm{i}(\varphi + 2\pi m)$$

Finally, the Q function takes the form of cyclic Gaussian (describing, in particular, the formation of the Schrödinger cats on the sphere)

$$Q_\mu(\vartheta, \varphi) = \frac{2^{2S}}{\sqrt{1+s^2\mu^2}} \left(\frac{|\xi|}{1+|\xi|^2}\right)^{2S} \left| \sum_{m=-\infty}^{\infty} \mathrm{e}^{(b_m^2/4a)} \right|^2 \tag{1.37}$$

Let us note that $|a| \ll 1$ for typical values of the parameter μ. With this assumption, the maximally squeezed state corresponds to a situation where each Gaussian in Equation 1.37 is very narrow. In this case, the dominant contribution to the Equation 1.37 comes from the $m = 0$ term. In this case, the approximate expression for the Q function takes the form

$$Q_\mu(\vartheta, \varphi) = \frac{2^A}{\sqrt{1+A^2\mu^2/4}} \left(\frac{|\xi|}{1+|\xi|^2}\right)^A \exp\left[\frac{A((\ln|\xi|)^2 - \varphi^2) + A^2\mu\varphi \ln|\xi|}{4(1+A^2\mu^2/4)}\right] \tag{1.38}$$

In particular, the coherent states correspond to the value 0 of the parameter μ

$$Q_0(\vartheta, \varphi) = 2^A \left(\frac{|\xi|}{1+|\xi|^2}\right)^A \exp\left[-\frac{A}{4}\left(\varphi^2 - (\ln|\xi|)^2\right)\right] \tag{1.39}$$

Recalling that in the limit $A \gg 1$, relative fluctuations in the direction perpendicular to $\mathbf{n} = (1, 0, 0)$ are of order $\sim A^{-1/2}$, we can expand ξ in series of ς, $|\varsigma| \ll 1$, where $\vartheta \sim \pi/2 - \varsigma$, thereby obtaining an expression for the projection of the Q function onto the tangent plane (φ, ς):

$$Q_0(\varphi, \varsigma) = \exp\left[-\frac{A}{4}\left(\varphi^2 + \varsigma^2\right)\right] \tag{1.40}$$

with the plane centered on the x axis.

The corresponding level curves for $Q_\mu(\vartheta, \varphi)$ take the form of ellipses for values of μ that are not very large, i.e. the initial coherent state becomes squeezed in a certain direction. For larger values of μ, these ellipses become twisted and deformed due to quantum self-interference effect (appearance of Schrödinger cat states on the sphere [18]). This deformation and deviation from the elliptical shape put limits to the maximum possible squeezing that can be achieved in the system with a given number of atoms subjected by the transformation (1.26).

An approximate expression for the Q function on the tangent (φ, ς) plane can be obtained by expanding $|\xi|$ in series of ς, giving

$$Q_\mu(\varsigma, \varphi) = \frac{1}{\sqrt{1 + A^2\mu^2/4}} \exp\left[-A\frac{\varsigma^2\left(2 + A^2\mu^2\right) + 2A\mu\varphi\varsigma + 2\varphi^2}{8\left(1 + A^2\mu^2/4\right)}\right] \tag{1.41}$$

The major axis of the ellipse makes an angle

$$\tan 2\phi = \frac{2}{A\mu}$$

with the φ axes. In the rotated frame,

$$\varphi' = \varphi\cos\phi + \varsigma\sin\phi$$

$$\varsigma' = \varsigma\cos\phi - \varphi\sin\phi$$

the Q function takes the form

$$Q_\mu(\varsigma', \varphi') = \frac{1}{\sqrt{A^2\mu^2/4 + 1}} \exp\left[-\frac{A}{8}\left(\frac{\varphi'^2}{a^2} + \frac{\varsigma'^2}{b^2}\right)\right]$$

where

$$a^2 = \left(2 - \frac{A\mu}{\sqrt{A^2\mu^2/4 + 1}}\right)^{-1}, \quad b^2 = \left(2 + \frac{A\mu}{\sqrt{A^2\mu^2/4 + 1}}\right)^{-1}$$

For large values of μ, $a^2 \to A^2\mu^2/4$, $b^2 \to 1/4$, exhibiting growing squeezing – beyond the maximum value (Equation 1.33). The above equations show that the tangent plane description of the atomic squeezing fails for large values of μ.

Another transformation that generates squeezing when applied to atomic coherent states is of the form

$$V_s(\mu) = \exp(\mu S_+^2 - \mu^* S_-^2)$$

Unfortunately, calculations for this transformation are not as simple as for the transformation (1.26). However, in the limit of small μ, we can use the Taylor expansion and find the transformed collective atomic operators up to the second order in μ with a consecutive calculation of fluctuations in the tangent plane.

1.5
Atoms with $n > 2$ Energy Levels

This section is devoted to the description of atomic systems with more than two energy levels [19]. First, we introduce a general formalism for a collection of A atoms, each atom containing n energy levels. We assume that the atoms are distinguishable, so the state vector describing the system is symmetric under permutation of indices labeling the individual atoms. We then specialize the discussion to systems with three energy levels, and then we consider a particular case of systems with three energy levels.

1.5.1
Systems with n Energy Levels

Let us consider a single atom with n energy levels. Vectors $|j\rangle$, $j = 1, 2, \ldots, n$, describing states with energies E_j (bare states) form an orthonormal basis, $\langle k|j\rangle = \delta_{jk}$, $\sum_{j=1}^{n} |j\rangle\langle j| = I$, in the n-dimensional Hilbert space for this single atom. We represent the states $|j\rangle$ as column vectors in the n-dimensional linear space

$$|1\rangle \to \begin{bmatrix} 0 \\ \cdots \\ 0 \\ \cdots \\ 1 \end{bmatrix}, \quad |j\rangle \to \begin{bmatrix} 0 \\ \cdots \\ 1_j \\ \cdots \\ 0 \end{bmatrix}, \quad |n\rangle \to \begin{bmatrix} 1 \\ \cdots \\ 0 \\ \cdots \\ 0 \end{bmatrix} \tag{1.42}$$

It is easy to see that a parameterization, similar to Equation 1.2, of an arbitrary state $|\psi\rangle$ has the form

$$|\psi\rangle \to \begin{bmatrix} e^{i\varphi_1} \cos \dfrac{\vartheta_1}{2} \\ e^{i\varphi_2} \sin \dfrac{\vartheta_1}{2} \cos \dfrac{\vartheta_2}{2} \\ \cdots \\ \cdots \\ e^{i\varphi_{n-1}} \sin \dfrac{\vartheta_1}{2} \cdots \sin \dfrac{\vartheta_{n-2}}{2} \cos \dfrac{\vartheta_{n-1}}{2} \\ \sin \dfrac{\vartheta_1}{2} \cdots \sin \dfrac{\vartheta_{n-1}}{2} \end{bmatrix} \tag{1.43}$$

where the phase of the ground state is chosen to be zero for simplicity. Clearly, the transformations preserving the norm of the state vector (1.43) are elements of the group $U(n)$.

The free Hamiltonian of the system has the form

$$H_0 = \sum_{j=1}^{n} E_j |j\rangle\langle j| \tag{1.44}$$

The configuration of the energy levels may be arbitrary, but it is convenient to choose the energies in such a way that $E_j \leq E_{j+1}$. Operators that generate transitions between levels are $s^{ij} = |j\rangle\langle i|$, $(j \neq i)$, $i, j = 1, \ldots, n$ and satisfy the following commutation relations:

$$\left[s^{ij}, s^{lk}\right] = \delta_{ik} s^{lj} - \delta_{jl} s^{ik} \tag{1.45}$$

which are those for the generators of the $U(n)$ algebra (see Appendix 11.1).

Let us recall that any element of the $U(n)$ group can be represented as the product of an element of the $su(n)$ group by an element of the $U(1)$ group. Obviously the element of the $U(1)$ group is related to a global phase of the system state. To exclude this phase, we use the traceless combinations

$$s_z^{jj+1} = \frac{1}{2}\left(|j+1\rangle\langle j+1| - |j\rangle\langle j|\right), \quad j = 1, \ldots, n-1$$

These operators are interpreted as atomic inversions between levels $j+1$ and j. This interpretation is convenient considering the ordering of our energy levels. Obviously, there are $n-1$ independent operators s_z^{jj+1}. The projectors $|j\rangle\langle j|$ are expressed in terms of s_z^{jj+1} as

$$s^{jj} = |j\rangle\langle j| = \frac{1}{n} + 2\sum_{k=1}^{j-1} s_z^{k\,k+1} - \frac{2}{n}\sum_{k=1}^{n-1}\left(n-k\right) s_z^{k\,k+1} \tag{1.46}$$

Thus, the free Hamiltonian (1.44) in terms of the operators s_z^{jj+1} has the form

$$H_0 = 2\sum_{k=1}^{n-1} s_z^{k\,k+1}\left[Ek - \sum_{j=1}^{k} E_j\right] + E \tag{1.47}$$

where E is the "energy of the center of the spectrum", which is defined as $E = \frac{1}{n}\sum_{j=1}^{n} E_j$. The operators s_z^{jj+1}, $s^{ij} = |i\rangle\langle j|$, $j \neq i$, $i, j = 1, \ldots, n$ form the n-dimensional representation of the $su(n)$ algebra.

In the case of A identical atoms with n energy levels, the collective operators

$$S^{ij} = \sum_{a=1}^{A} |a, i\rangle\langle a, j|, \quad i, j = 1, \ldots, n \tag{1.48}$$

are introduced where $|a, i\rangle$ is a state of the ath atom. The operators S^{ij} obviously satisfy commutation relations identical to those of the individual atomic operators of Equation 1.45. Thus the problem is reduced to the construction of the irreducible representations for the $su(n)$ group.

If the atoms are *identical* (the Hamiltonian is invariant under permutations of atoms), it is sufficient to consider only the symmetric representation. A useful method for obtaining the matrix elements of operators acting in the space of this representation of $su(n)$ is the so-called Schwinger construction. Let us introduce a set of auxiliary creation and annihilation operators satisfying the usual bosonic commutation relations:

$$\left[b_i, b_j^\dagger\right] = \delta_{ij}, \quad \left[b_i, b_j\right] = \left[b_i^\dagger, b_j^\dagger\right] = 0$$

We may think of b_i and $b_i^\dagger$ as the creation and annihilation operators for the excitation at the ith atomic level ($i = 1, 2, \ldots, n$). In terms of b_i and $b_j^\dagger$, the collective atomic operators have the form

$$S^{ij} = b_j^\dagger b_i, \quad i, j = 1, 2, \ldots, n \tag{1.49}$$

It is directly checked that these operators satisfy the commutation relations of the $u(n)$ algebra (1.45). The collective operators (1.49) act on the Hilbert space of n harmonic oscillators with the basis

$$|k_1, k_2, \ldots, k_n\rangle = |k_1\rangle \otimes |k_2\rangle \otimes \cdots \otimes |k_n\rangle$$

such that

$$b_i|k_1, \ldots, k_i, \ldots, k_n\rangle = \sqrt{k_i}\,|k_1, \ldots, k_i - 1, \ldots, k_n\rangle$$

It can easily be seen that the following operator commutes with all the operators of the algebra:

$$\hat{A} = \sum_{j=1}^{n} S^{jj}, \quad \left[\hat{A}, S^{ij}\right] = 0$$

The eigenvalue of this operator $\hat{A}$ is just $k_1 + k_2 + \cdots + k_n = A$. Here, k_i represents the number of atoms at the ith energy level and A is the total number of atoms. The free atomic Hamiltonian takes on the form

$$H_0 = \sum_{i=1}^{n} E_i S^{ii} = \sum_{i=1}^{n} E_i b_i^\dagger b_i$$

The atomic inversion operators are now introduced as

$$S_z^{ij} = \frac{1}{2}\left(S^{jj} - S^{ii}\right) = \frac{1}{2}\left(b_j^\dagger b_j - b_i^\dagger b_i\right)$$

and the transition operators $S_\pm^{ji}$, ($j > i$)

$$S_+^{ij} = S^{ij} = b_j^\dagger b_i, \qquad S_-^{ij} = \left(S_+^{ij}\right)^\dagger = S^{ji} = b_i^\dagger b_j$$

It is easy to find the matrix elements of the operators $S_\pm^{ij}$, Using Equation 1.49

$$\begin{aligned}
S_+^{ij}|k_1, \ldots, k_i, \ldots, k_j, \ldots, k_n\rangle &= \sqrt{k_i(k_j+1)}|k_1, \ldots, k_i - 1, \ldots, k_j + 1, \ldots, k_n\rangle, \\
S_-^{ij}|k_1, \ldots, k_i, \ldots, k_j, \ldots, k_n\rangle &= \sqrt{k_j(k_i+1)}|k_1, \ldots, k_i + 1, \ldots, k_j - 1, \ldots, k_n\rangle, \\
S_z^{ij}|k_1, \ldots, k_i, \ldots, k_j, \ldots, k_n\rangle &= \frac{1}{2}(k_j - k_i)|k_1, \ldots, k_i, \ldots, k_j, \ldots, k_n\rangle
\end{aligned} \tag{1.50}$$

The operators $\{S_z^{ij}, S_+^{ij}, S_-^{ij}\}_{i \neq j}$ span a representation of the $su(n)$ algebra. Note that the operators S_z^{ij} and $S_\pm^{ij}$ form $su(2)$ subalgebras:

$$\left[S_z^{ij}, S_\pm^{ij}\right] = \pm S_\pm^{ij} \tag{1.51}$$

1.5.2
Systems with Three Energy Levels

Let us consider a particular case of systems with three energy levels. A pure state is a superposition of bare states $|j\rangle, j = 1, 2, 3$,

$$|\psi\rangle = \cos\vartheta_1|1\rangle + e^{i\varphi_1}\sin\vartheta_1\cos\vartheta_2|2\rangle + e^{i\varphi_2}\sin\vartheta_2\sin\vartheta_1|3\rangle$$

where $0 \le \varphi_{1,2} \le 2\pi,\ \ 0 \le \vartheta_{1,2} \le \pi/2$. In terms of the diagonal projectors $|j\rangle\langle j|$, the free Hamiltonian for a single atom has the form

$$H_0 = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| + E_3|3\rangle\langle 3|$$

so that $|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| = I$.

We suppose that the transitions in atomic systems only take place in the dipole approximation, that is, there are no transitions between levelswith the same parity. Thus, there are only three possible configurations for systems with three energy levels: cascade (Ξ), lambda (Λ), and (V) configurations. Enumerating energy levels in such a way that $E_1 \le E_2 \le E_3$, we see that the transitions allowed in the Ξ system are $1 \leftrightarrow 2 \leftrightarrow 3$; in the Λ system, $1 \leftrightarrow 3 \leftrightarrow 2$; and in the V system $3 \leftrightarrow 1 \leftrightarrow 2$. Inversion operators relevant to each physically different configuration can be constructed without difficulty:

1. For the Ξ-type system the inversion operators are defined as

$$s_z^{12} = \frac{1}{2}\left(|2\rangle\langle 2| - |1\rangle\langle 1|\right), \quad s_z^{23} = \frac{1}{2}\left(|3\rangle\langle 3| - |2\rangle\langle 2|\right)$$

and the free Hamiltonian takes on the form

$$H_0 = \frac{2}{3}\left(E_2 + E_3 - 2E_1\right) s_z^{12} + \frac{2}{3}\left(2E_3 - E_1 - E_2\right) s_z^{23} + E \tag{1.52}$$

Here, the energy of the center of a spectrum E is equal to

$$E = \frac{1}{3}\left(E_1 + E_2 + E_3\right)$$

2. For the Λ-type system, the inversion operators are defined as

$$s_z^{13} = \frac{1}{2}\left(|3\rangle\langle 3| - |1\rangle\langle 1|\right), \quad s_z^{23} = \frac{1}{2}\left(|3\rangle\langle 3| - |2\rangle\langle 2|\right)$$

and the free Hamiltonian takes on the form

$$H_0 = \frac{2}{3}\left(E_2 + E_3 - 2E_1\right) s_z^{13} + \frac{2}{3}\left(E_1 + E_3 - 2E_2\right) s_z^{23} + E \tag{1.53}$$

3. For the V-type system the inversion operators are defined as

$$s_z^{12} = \frac{1}{2}\left(|2\rangle\langle 2| - |1\rangle\langle 1|\right), \quad s_z^{13} = \frac{1}{2}\left(|3\rangle\langle 3| - |1\rangle\langle 1|\right)$$

and the free Hamiltonian takes on the form

$$H_0 = \frac{2}{3}\left(2E_2 - E_1 - E_3\right) s_z^{12} + \frac{2}{3}\left(2E_3 - E_1 - E_2\right) s_z^{13} + E \tag{1.54}$$

The transition operators in the system are given by Equations 1.7 and 1.45. Some of the transitions are absent in the Ξ, Λ, or V configurations. Explicitly, the transition operators for these cases are

$$\Xi : |2\rangle\langle 1|,\ |3\rangle\langle 2|,\ h.c.$$

$$\Lambda : |3\rangle\langle 1|,\ |3\rangle\langle 2|,\ h.c.$$

$$V : |2\rangle\langle 1|,\ |3\rangle\langle 1|,\ h.c$$

We can immediately write, using Equations 1.48, the operators describing a collection of A indistinguishable atoms with three energy levels. There are still three configurations, and appropriate operators are simply obtained by substituting $s_z^{jk} \leftrightarrow S_z^{jk}$ and $E \to AE$. Then the atomic transition operators are

$$\Xi : S_{\pm}^{12}, \quad S_{\pm}^{23}$$

$$\Lambda : S_{\pm}^{13}, \quad S_{\pm}^{23}$$

$$V : S_{\pm}^{12}, \quad S_{\pm}^{13}$$

The matrix realization of each $S_z^{jk}, S_{\pm}^{jk}$ for each configuration form a symmetric irreducible representation of the $su(3)$ algebra of the dimension $(A+1)(A+2)/2$. The operators $\{S_z^{jk},\ S_+^{jk},\ S_-^{jk}\}$ form the $su(2)$ subalgebras (1.51) and all the S_z^{jk} operators commute. All the other commutation relations and the matrix elements of the collective operators in symmetric representations are easily obtained using the Schwinger representation for the collective atomic operators.

1.6
Problems

1.1 Show that the Von Neumann entropy of a two-level system, defined as $S = -\mathrm{Tr}(\rho \log \rho)$ is a monotonous function of the purity of the atomic state, $\mathcal{P} = \mathrm{Tr}(\rho^2)$ in the interval $1/2 \leq \mathcal{P} \leq 1$. Hint: prove that $S = -\lambda_1 \log \lambda_1 - \lambda_2 \log \lambda_2$, where $\lambda_{1,2} = (1 \pm \sqrt{2\mathcal{P} - 1})/2$.

1.2 For squeezed states in the case of two atoms, $A = 2$, find the state $|\psi_0\rangle = U_s(\mu_0)|\pi/2, 0\rangle$ and direction δ_0 for which the fluctuation of the projection of vector $\mathbf{S}$ is zero (i.e. $|\psi_0\rangle$ is the eigenstate of the operator $s(\delta_0)$). *Answer:*

$$|\psi_0\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ -i\sqrt{2} \\ 1 \end{bmatrix}, \quad \delta_0 = \frac{3\pi}{4}$$

1.3 Prove that for squeezed states in the case of many atoms, $A \gg 1$ and $|\mu| \ll 1$, the following approximation can be made for $\langle \Delta s^2(\mu) \rangle$ in Equation 1.27,

$$\langle \Delta s^2(\mu) \rangle \approx \frac{A}{4}\left(\frac{1}{4\alpha^2} + \frac{2}{3}\beta^2\right)$$

where $\alpha = (A/2)\mu$ and $\beta = (A/2)\mu^2$, and its minimum value,

$$\langle \Delta s^2(\mu) \rangle_{\mathrm{min}} \approx \frac{1}{2}\left(\frac{A}{6}\right)^{1/3} \tag{1.55}$$

is achieved when

$$\mu_0 = \frac{24^{1/6}}{2}\left(\frac{A}{2}\right)^{-2/3} \tag{1.56}$$

2
Atomic Dynamics

In this chapter, we consider the simplest examples of the two-level system dynamics in an external classical field. We start with the spin dynamics in a constant magnetic field. Then we consider the resonance dynamics of a two-level atom in an oscillating electric field for different cases of the field polarization, and, in doing so, justify the rotating wave approximation (RWA). We also discuss two-level system dynamics beyond the RWA. Finally, we consider examples of the collective dynamics of an ensemble of two-level systems in general time-dependent fields.

2.1
Spin in a Constant Magnetic Field

As we have already mentioned, a two-level atom can be described in terms of isotopic spin. Thus, the dynamics of a two-level atom in a classical external field is equivalent to the dynamics of a spin 1/2 particle in a corresponding magnetic field.

We begin with the simplest example: the evolution in a *constant* magnetic field. Let us denote the magnetic field vector by $\boldsymbol{\mathcal{H}}$ and the magnetic moment of the spin by $\boldsymbol{\mu} = \mu\boldsymbol{\sigma}$, where $\boldsymbol{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$. The Hamiltonian has the form $H = -\left(\boldsymbol{\mu} \cdot \boldsymbol{\mathcal{H}}\right)$, or explicitly,

$$H = -\mu\left(\boldsymbol{\sigma} \cdot \boldsymbol{\mathcal{H}}\right) = -\mu \begin{bmatrix} \mathcal{H}_z & \mathcal{H}_x - \mathrm{i}\mathcal{H}_y \\ \mathcal{H}_x + \mathrm{i}\mathcal{H}_y & -\mathcal{H}_z \end{bmatrix} \tag{2.1}$$

The solution of the Schrödinger equation ($\hbar = 1$),

$$\mathrm{i}\,\partial_t|\psi(t)\rangle = H|\psi(t)\rangle$$

has the form

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle\,, \quad U(t) = \exp(-\mathrm{i}Ht)$$

where $U(t)$ is the evolution operator. In this case, the exponential $\exp(-\mathrm{i}Ht)$ can be directly evaluated by expanding it in a Taylor series. Since $H^2 = \Omega^2 I$, where

$$I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Omega = \mu\sqrt{\mathcal{H}_x^2 + \mathcal{H}_y^2 + \mathcal{H}_z^2}$$

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we obtain the system evolution operator in the constant field:

$$U(t) = 1 - \mathrm{i}tH - \frac{t^2}{2!}H^2 + \cdots = I\cos\Omega t - \frac{\mathrm{i}H}{\Omega}\sin\Omega t$$

$$= \begin{bmatrix} \cos\Omega t + \dfrac{\mathrm{i}\mu\mathcal{H}_z}{\Omega}\sin\Omega t & \mu\dfrac{\mathrm{i}\mathcal{H}_x + \mathcal{H}_y}{\Omega}\sin\Omega t \\ \mu\dfrac{\mathrm{i}\mathcal{H}_x - \mathcal{H}_y}{\Omega}\sin\Omega t & \cos\Omega t - \dfrac{\mathrm{i}\mu\mathcal{H}_z}{\Omega}\sin\Omega t \end{bmatrix} \tag{2.2}$$

It is obvious that $U(t)^\dagger = U(-t)$. The evolution of the system initially prepared in an arbitrary state $|\mathrm{in}\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, $|\alpha|^2 + |\beta|^2 = 1$, is described by the state vector $|\Psi(t)\rangle = U(t)\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. The eigenvalues of the Hamiltonian Equation 2.1 are $\pm\Omega$,

$$H|\pm\rangle = \pm\Omega|\pm\rangle$$

and the corresponding eigenvectors have the form

$$|\pm\rangle = \frac{\mu}{\sqrt{2\Omega(\Omega \mp \mu\mathcal{H}_z)}}\begin{bmatrix} \mathcal{H}_x - \mathrm{i}\mathcal{H}_y \\ -\mathcal{H}_z \pm \dfrac{\Omega}{\mu} \end{bmatrix}$$

2.2
A Two-level Atom in a Linearly Polarized Field

Let us now suppose that the atom has only one electron and is plunged in a time-varying electric field, as in the model discussed in the previous chapter. The Hamiltonian describing the interaction of the atom with the electric field has the form $H_{\text{int}} = -(\mathbf{d}\cdot\boldsymbol{\mathcal{E}})$, where $\mathbf{d}$ is the electric dipole operator (Section 1.1). We have neglected the interaction between the induced magnetic field and the magnetic moment of the atom. In the case of an atom in a linearly polarized field, we choose the x axis along the direction of the electric field:

$$\boldsymbol{\mathcal{E}} = \left(\mathcal{E}_0\cos\omega_f t,\ 0,\ 0\right)$$

The interaction Hamiltonian takes the form (see Equation 1.6)

$$H_{\text{int}} = -\mathcal{E}_0\cos\omega_f t\left(\sigma_x\,\mathrm{Re}\,\mathrm{d}_x - \sigma_y\,\mathrm{Im}\,\mathrm{d}_x\right) = 2(g\sigma_+ + g^*\sigma_-)\,\cos\omega_f t \tag{2.3}$$

where $\mathrm{d}_x = \mathrm{e}\langle 1|x|0\rangle$ and $g = -\mathrm{d}_x\mathcal{E}_0/2$.

2.2.1
The Rotating Wave Approximation

Although the Hamiltonian Equation 2.3 has a very simple form, a further approximation is necessary to obtain an analytic solution. To the time-dependent ket, we apply an extra transformation generated by the operator σ_z to obtain the ket $|\psi(t)\rangle_R$ defined as

$$|\psi(t)\rangle_R = \mathrm{e}^{\mathrm{i}\omega_f\sigma_z t/2}|\psi(t)\rangle$$

This is a transformation to a rotating frame in the pseudospin space, leading to the transformed Schrödinger equation ($\hbar = 1$)

$$\mathrm{i}\partial_t|\psi\rangle_R = -\frac{\omega_f\sigma_z}{2}|\psi\rangle_R + \mathrm{e}^{\mathrm{i}\omega_f\sigma_z t/2} H\,\mathrm{e}^{-\mathrm{i}\omega_f\sigma_z t/2}|\psi\rangle_R \equiv H_R|\psi\rangle_R$$

where

$$H = H_0 + H_{\text{int}}$$

$$H_0 = \frac{\omega_{\text{a}}}{2}\,\sigma_z\,,\quad \omega_{\text{a}} = E_1 - E_0$$

Using the transformation

$$\mathrm{e}^{\mathrm{i}\sigma_z\tau/2}\sigma_\pm\,\mathrm{e}^{-\mathrm{i}\sigma_z\tau/2} = \sigma_\pm\,\mathrm{e}^{\pm\mathrm{i}\tau}$$

the Hamiltonian H_R becomes

$$H_R = \frac{\omega_{\text{a}} - \omega_f}{2}\,\sigma_z + g\sigma_+ + g^*\sigma_- + g^*\sigma_+\,\mathrm{e}^{\mathrm{i}2\omega_f t} + g\sigma_-\,\mathrm{e}^{-\mathrm{i}2\omega_f t}$$

and the equations of motion for the components of vector $|\psi\rangle_R = \begin{bmatrix} c_1 \\ c_0 \end{bmatrix}$ are

$$\mathrm{i}\dot{c}_1 = \frac{\Delta}{2}\,c_1 + gc_0 + g^*\mathrm{e}^{\mathrm{i}2\omega_f t}c_0,\quad \mathrm{i}\dot{c}_0 = -\frac{\Delta}{2}\,c_0 + g^*c_1 + g\,\mathrm{e}^{\mathrm{i}2\omega_f t}c_1$$

where $\Delta = \omega_{\text{a}} - \omega_f$ is the so-called detuning parameter. In typical optical systems, the field frequency ω_f is such that the term $\exp\left(2\mathrm{i}\omega_f t\right)$ oscillates rapidly, and we have, in addition, $|g| \ll \omega_{\text{a}}$. Thus, after averaging over rapid oscillations, the equations of motions acquire the stationary form

$$\mathrm{i}\dot{c}_1 = \frac{\Delta}{2}\,c_1 + gc_0,\quad \mathrm{i}\dot{c}_0 = -\frac{\Delta}{2}\,c_0 + g^*c_1$$

This corresponds to the effective Hamiltonian

$$H_R^{\text{RWA}} = \frac{\Delta}{2}\,\sigma_z + g\sigma_+ + g^*\sigma_- \tag{2.4}$$

This approximation, in which rapidly oscillating terms are neglected, is called the *rotating wave approximation* (RWA). In this approximation we can, *from the beginning*, replace the Hamiltonian (2.3) by the effective Hamiltonian

$$H^{\text{RWA}} = \frac{\omega_{\text{a}}}{2}\,\sigma_z + g\sigma_+\,\mathrm{e}^{-\mathrm{i}\omega_f t} + g^*\sigma_-\,\mathrm{e}^{\mathrm{i}\omega_f t} \tag{2.5}$$

Note that under the RWA, the rotation around the z axis in the pseudospin space is equivalent to the change of phase of the external field in the Hamiltonian (2.3) and leads to the change of phase of the coupling constant g.

The Hamiltonian (2.4) has the same form as the Hamiltonian for a spin in the external magnetic field. All we need is to replace

$$-\mu\mathcal{H}_z \longrightarrow \frac{\Delta}{2},\quad -\mu\mathcal{H}_x \longrightarrow \operatorname{Re} g,\quad \mu\mathcal{H}_y \longrightarrow \operatorname{Im} g$$

Thus, in the RWA, the evolution operator is obtained directly from Equation 2.2:

$$U(t) = \begin{bmatrix} \cos\Omega t - \dfrac{i\Delta}{2\Omega}\sin\Omega t & -\dfrac{ig}{\Omega}\sin\Omega t \\ \dfrac{ig^*}{\Omega}\sin\Omega t & \cos\Omega t + \dfrac{i\Delta}{2\Omega}\sin\Omega t \end{bmatrix}$$

where $\Omega = \sqrt{\Delta^2/4 + |g|^2}$ is called the it Rabi frequency. The eigenvalues of the Hamiltonian are $\pm\Omega$, and the eigenvectors have the form

$$|\pm\rangle = \frac{1}{\sqrt{\Omega(2\Omega \mp \Delta)}} \begin{bmatrix} g \\ -\dfrac{\Delta}{2} \pm \Omega \end{bmatrix}$$

2.3
A Two-level Atom in a Circularly Polarized Field

In the following, we consider again the case of atoms with only one electron. In fact, the index of the vectors of atomic basis (1.4) has three components: $|\Psi_n\rangle \to |\alpha, l, m\rangle$, where l is the angular momentum, m is the z component of the angular momentum, and the label alpha stands for all other quantum numbers required to completely specify the state (see, for example, [20]). Let us consider the transition between the states $|0\rangle = |\alpha, l, m\rangle$ and $|1\rangle = |\beta, l', m+1\rangle$. We find that the admissible values of l' are $l+1,\ l,\ l-1$.

The interaction Hamiltonian $H_{\mathrm{int}} = -(\mathbf{d}\cdot\boldsymbol{\mathcal{E}})$ for an atom with a circularly polarized field

$$\mathcal{E}_x = \mathcal{E}_0 \cos\omega_f t\,, \quad \mathcal{E}_y = \mathcal{E}_0 \sin\omega_f t\,, \quad \mathcal{E}_z = 0$$

takes the form

$$H_{\mathrm{int}} = -\mathcal{E}_0\big[\big(d_x\cos\omega_f t + d_y\sin\omega_f t\big)\sigma_+ + \big(d_x^*\cos\omega_f t + d_y^*\sin\omega_f t\big)\sigma_-\big]$$

where we have used the representation (1.6) for the dipole momentum operator, $d_x = \mathrm{e}\langle 1|x|0\rangle$ and $d_y = \mathrm{e}\langle 1|y|0\rangle$. It is shown below that

$$\langle 1|(x - iy)|0\rangle = 0 \tag{2.6}$$

which allows us to arrive at the following form of the interaction Hamiltonian:

$$H_{\mathrm{int}} = g\sigma_+\,\mathrm{e}^{-i\omega_f t} + g^*\sigma_-\,\mathrm{e}^{i\omega_f t} \tag{2.7}$$

where $g = -\mathcal{E}_0(d_x + id_y)/2$. It is worth noting that the interaction Hamiltonian (2.7) has exactly the same form as the effective Hamiltonian (2.5), although we are not using the RWA, so that the corresponding evolution operator is given by Equation 2.2.

Let us now prove the result in Equation 2.6. The state $|0\rangle = |\alpha, l, m\rangle$ has a wave function

$$\langle \mathbf{r}|\alpha, l, m\rangle = \Psi_{\alpha,l,m}(\mathbf{r}) = \psi_{\alpha,l}(r)\,Y_{lm}(\theta,\phi) \tag{2.8}$$

where r, θ, ϕ are polar coordinates. The wave function $\Psi_{\alpha,l,m}(\mathbf{r})$ is an eigenstate of the z component of the angular momentum operator L_z:

$$L_z\Psi_{\alpha,l,m}(\mathbf{r}) = m\Psi_{\alpha,l,m}(\mathbf{r}), \quad L_z = -\mathrm{i}(x\partial_y - y\partial_x)$$

Now, the functions $|\alpha, l, m\rangle$ form a basis for the Hilbert space of states describing the atom. On the other hand, the functions $x \pm \mathrm{i}y$ are also eigenfunctions of the operator L_z with the eigenvalues ± 1:

$$L_z(x \pm \mathrm{i}y) = -\mathrm{i}(x\partial_y - y\partial_x)(x \pm \mathrm{i}y) = \pm(x \pm \mathrm{i}y)$$

and, thus, the functions $f(\mathbf{r}) = (x \pm \mathrm{i}y)\Psi_{n,l,m}(\mathbf{r})$ are also eigenfunctions of L_z with the eigenvalues $m \pm 1$:

$$L_z f(\mathbf{r}) = \left[L_z(x \pm \mathrm{i}y)\right] \Psi_{\alpha,l,m}(\mathbf{r}) + (x \pm \mathrm{i}y) \left[L_z\Psi_{\alpha,l,m}(\mathbf{r})\right] = (m \pm 1)f(\mathbf{r})$$

This means that only the term with the value of the third index $m \pm 1$ appears in the expansion of the function $f(\mathbf{r})$ in the basis (2.8):

$$(x \pm \mathrm{i}y)\Psi_{\alpha,l,m}(\mathbf{r}) = \sum_{l'} \phi_{l'}(r) Y_{l',m\pm 1}(\theta, \phi) \tag{2.9}$$

Here $\phi_{l'}(r)$ are radial functions. Indeed, the angular momentum operators $L_{x,y,z}$ form the $su(2)$ algebra; the functions $Y_{lm}(\theta, \phi)$ form a basis in the irreducible $2l + 1$ dimension representation of this algebra (see Appendix 11.1); on the other hand, the functions $x + \mathrm{i}y$, $\sqrt{2}z$, and $x - \mathrm{i}y$ form a basis in the three-dimensional representation of the $su(3)$, which corresponds to $l = 1$; the function $f(\mathbf{r})$ belongs to the space of the tensor product of these two representations; this tensor product can be reduced to the sum of three irreducible parts: $l' = l + 1$, l or $l - 1$, (if $l > 0$); thus, only the terms $l' = l + 1$, l, $l - 1$ appear in the sum (2.9). Finally, to obtain matrix elements $\langle\beta, l', m'|(x \pm \mathrm{i}y)|\alpha, l, m\rangle$ we multiply Equation 2.9 by $\psi^*_{\beta,l'}(r)Y^*_{l'm'}(\theta, \phi)$, (from Equation 2.8) and integrate over θ, ϕ, and r. Since the spherical functions $Y_{lm}(\theta, \phi)$ are orthogonal, only the matrix elements

$$\langle\beta, l', m \pm 1|(x \pm \mathrm{i}y)|\alpha, l, m\rangle\,, \quad l' = l + 1,\ l,\ l - 1$$

are different from zero. If, for example, the second state of the atom is $|1\rangle = |\beta, l', m + 1\rangle$, then

$$\langle\beta, l', m + 1|(x - \mathrm{i}y)|\alpha, l, m\rangle = \langle\beta, l', m|(x + \mathrm{i}y)|\alpha, l, m + 1\rangle = 0$$

and we arrive at Equation 2.6. The matrix elements of the Hamiltonian H_{int} from Equation 2.7 are

$$\langle 1|H_{\text{int}}|0\rangle = \frac{-\mathrm{e}\mathcal{E}_0}{2}\langle\beta, l', m + 1|(x + \mathrm{i}y)|\alpha, l, m\rangle \mathrm{e}^{-\mathrm{i}\omega_f t} \equiv g\,\mathrm{e}^{-\mathrm{i}\omega_f t}$$

$$\langle 0|H_{\text{int}}|1\rangle = \frac{-\mathrm{e}\mathcal{E}_0}{2}\langle\beta, l', m|(x - \mathrm{i}y)|\alpha, l, m + 1\rangle \mathrm{e}^{\mathrm{i}\omega_f t} = g^*\mathrm{e}^{\mathrm{i}\omega_f t}$$

2.4
Evolution of the Bloch Vector

In general, the Hamiltonian of a two-level system has the form

$$H = \frac{1}{2}\left(\Omega_x \sigma_x + \Omega_y \sigma_y + \Omega_z \sigma_z\right) \tag{2.10}$$

For example, for a spin in a magnetic field, $\boldsymbol{\Omega} = -2\mu\boldsymbol{\mathcal{H}}$; or, for an atom in a linearly polarized electric field under the RWA

$$\boldsymbol{\Omega} = \left(2g\cos\omega t,\ 2g\sin\omega t,\ \omega_a\right) \tag{2.11}$$

For the case of the circularly polarized field, $\boldsymbol{\Omega}$ has the same form (2.11) without assuming the RWA. Note that in the last case the projections of the vectors of the field $\boldsymbol{\mathcal{E}}$ and of the "torque" $\boldsymbol{\Omega}$ onto the (x, y)plane move in the same manner [21]. The transformation to the rotating frame induces the following transformation of the matrices σ_x and σ_y:

$$\begin{aligned} e^{i\sigma_z\tau/2}\sigma_x e^{-i\sigma_z\tau/2} &= \sigma_x\cos\tau - \sigma_y\sin\tau \\ e^{i\sigma_z\tau/2}\sigma_y e^{-i\sigma_z\tau/2} &= \sigma_y\cos\tau + \sigma_x\sin\tau \end{aligned}$$

where $\tau = \omega t$. The Hamiltonian (2.10) and Equation 2.11 transform into

$$H = \frac{1}{2}\left(\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}\right), \quad \boldsymbol{\Omega} = (2g,\ 0,\ \omega_a) \tag{2.12}$$

The dynamics of a two-level atom in an external field can be described from another point of view. Let us find the equations of motion for the Heisenberg operators σ_j, $j = x, y, z$ with the Hamiltonian (2.10):

$$\begin{aligned} i\dot{\sigma}_x &= [\sigma_x, H] = i\left(\Omega_y\sigma_z - \Omega_z\sigma_y\right) \\ i\dot{\sigma}_y &= \left[\sigma_y, H\right] = i\left(\Omega_z\sigma_x - \Omega_x\sigma_z\right) \\ i\dot{\sigma}_z &= [\sigma_z, H] = i\left(\Omega_x\sigma_y - \Omega_y\sigma_x\right) \end{aligned}$$

One can rewrite this system in the matrix form:

$$\frac{d}{dt}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{bmatrix} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{bmatrix}$$

This system is linear and therefore, by introducing the Bloch vector as the average of the operators $\sigma_{x,y,z}$, one can write

$$\frac{d}{dt}\langle\boldsymbol{\sigma}\rangle = \widehat{\Omega}\,\langle\boldsymbol{\sigma}\rangle \tag{2.13}$$

where

$$\langle \boldsymbol{\sigma} \rangle = \begin{bmatrix} \langle \sigma_x \rangle \\ \langle \sigma_y \rangle \\ \langle \sigma_z \rangle \end{bmatrix}, \quad \widehat{\Omega} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$$

The Bloch Equation 2.13 can be written in the classical form using the cross product of the Bloch vector $\langle \boldsymbol{\sigma} \rangle$ and the torque vector $\boldsymbol{\Omega}$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{\sigma} \rangle = [\boldsymbol{\Omega} \times \langle \boldsymbol{\sigma} \rangle], \quad \boldsymbol{\Omega} = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$$

To solve the Bloch Equation 2.13, note that $\widehat{\Omega}\,\boldsymbol{\Omega} = 0$, and, so the vector $\boldsymbol{\Omega}$ defines the rotation axis (which does not change during the evolution). Squaring the operator $\hat{\Omega}$, we obtain

$$\widehat{\Omega}^2 = -\Omega^2 + \boldsymbol{\Omega} \otimes \boldsymbol{\Omega}^T, \quad \Omega = |\boldsymbol{\Omega}|$$

where $\boldsymbol{\Omega} \otimes \boldsymbol{\Omega}^T$ is the projection operator onto the vector $\boldsymbol{\Omega}$:

$$\boldsymbol{\Omega} \otimes \boldsymbol{\Omega}^T = \begin{bmatrix} \Omega_x^2 & \Omega_x\Omega_y & \Omega_x\Omega_z \\ \Omega_y\Omega_x & \Omega_y^2 & \Omega_y\Omega_z \\ \Omega_z\Omega_x & \Omega_z\Omega_y & \Omega_z^2 \end{bmatrix}$$

Expanding the Bloch equation evolution operator $U(t) = \exp(t\widehat{\Omega})$ in a Taylor series and using the equations above, we have

$$U = \mathrm{e}^{t\widehat{\Omega}} = \frac{\boldsymbol{\Omega} \otimes \boldsymbol{\Omega}^T}{\Omega^2} + \left(1 - \frac{\boldsymbol{\Omega} \otimes \boldsymbol{\Omega}^T}{\Omega^2}\right) \cos \Omega t + \widehat{\Omega}\, \frac{\sin \Omega t}{\Omega}$$

Of course, this evolution operator is equivalent to Equation 2.2 and involves the same Rabi frequency Ω. The difference is that, whereas we used earlier the *su*(2) algebra and its representation in the complex two-dimensional space, here we have found the solution in the real three-dimensional space, a method that presents an attractive geometrical picture.

2.5
Dynamics of the Two-level Atom without the RWA

Let us consider again the interaction of a two-level atom with the linearly polarized field (1.5). The Hamiltonian takes the form

$$H = \frac{\omega}{2}\, \sigma_z + g \sigma_x \cos \omega t \tag{2.14}$$

This Hamiltonian is periodic in time, with a period $T = 2\pi/\omega$, $H(t+T) = H(t)$. Thus, the Schrödinger equation is reduced to a system of two coupled first-order ordinary differential equations, with coefficients periodic in time and having the same period as the Hamiltonian. For such a system, it is known that there exist solutions of the form

$$|\Psi_\beta(t)\rangle = |\phi_\beta(t)\rangle\, \mathrm{e}^{-\mathrm{i}\lambda_\beta t}, \quad \beta = 0, 1$$

such that the vectors $|\phi_\beta(t)\rangle$ are periodic in time:

$$|\phi_\beta(t+T)\rangle = |\phi_\beta(t)\rangle$$

The existence of periodic (up to the phase factor) solutions is known as *Floquet's theorem*. It allows us to expand the functions $\phi_\beta(t)$ in Fourier series and rewrite the Schrödinger equation as an equation for the corresponding Fourier coefficients. This leads to an effective Hamiltonian in the infinite-dimensional space of Fourier coefficients. To solve the latter kind of problem, various perturbative approaches can be employed [22, 23].

Here we use the Lie-type perturbation theory [24, 25], which will allow us later in this book to treat more complicated quantum optical Hamiltonians. We start by rewriting the Hamiltonian in a time-independent form in the spirit of Floquet's theory. Then, using the Lie-type transformations (see Appendix 11.4) we reveal the possible resonant transitions that can be observed in the system. Actually, by slightly generalizing the problem, we consider here a collection of A two-level atoms driven by a linearly polarized monochromatic field of frequency Ω. The initial Hamiltonian has the form

$$H = \omega S_z + g\,(S_+ + S_-)\cos\Omega t \tag{2.15}$$

Here, $S_{+,-,z}$ are collective spin operators. If $S_\pm \to \sigma_\pm$, $S_z \to \sigma_z/2$, the Hamiltonian is reduced to the one-atom case. First of all, let us transform away the time dependence of the Hamiltonian (2.15). To achieve this, we extend the Hilbert space of our system by introducing the following operators (generators of Euclidean algebra):

$$[E_0, E] = -E, \quad \left[E_0, E^\dagger\right] = E^\dagger, \quad \left[E, E^\dagger\right] = 0 \tag{2.16}$$

(The name of the algebra follows from the fact that the operators E_0, $E_x = E^\dagger + E$, $E_y = -\mathrm{i}(E^\dagger - E)$ generate the group of motions of the Euclidean plane.) These operators act in a Hilbert space spanned by eigenstates of the operator $E_0 = E_0^\dagger$:

$$E_0|m\rangle = m|m\rangle, \quad m = \cdots -1, 0, 1, \ldots \tag{2.17}$$

so that in the basis (2.17), the operators E, $E^\dagger$ act as raising and lowering operators, respectively:

$$E|m\rangle = |m-1\rangle, \quad E^\dagger|m\rangle = |m+1\rangle$$

In Chapter 5, we see that Euclidean algebra naturally appears as a semiclassical limit of the Heisenberg algebra, when the mean number of photons $\overline{n} = \langle a^\dagger a\rangle$ is large

$$\mathrm{e}^{\mathrm{i}\phi} = \frac{1}{\sqrt{aa^\dagger}}\,a \longrightarrow E, \quad \mathrm{e}^{-\mathrm{i}\phi} = a^\dagger\frac{1}{\sqrt{aa^\dagger}} \longrightarrow E^\dagger, \quad a^\dagger a - \overline{n} \longrightarrow E_0$$

Here, E and $E^\dagger$ are field phase operators, E_0 is the shifted photon-number operator, and $m = n - \overline{n}$. The commutation relations (2.16) are approximately satisfied if $1 \ll \Delta n \ll \overline{n}$. Also, the eigenstates of E (phase states) can formally be introduced as

$$|\theta\rangle = \lim_{N\to\infty} \frac{1}{\sqrt{2N+1}} \sum_{m=-N}^{N} \mathrm{e}^{-\mathrm{i}\theta m}|m\rangle_j, \;\; E|\theta\rangle = \mathrm{e}^{\mathrm{i}\theta}|\theta\rangle, \;\; E^\dagger|\theta\rangle = \mathrm{e}^{-\mathrm{i}\theta}|\theta\rangle$$

where $|m\rangle$ are the basis states 2.17. We now consider the rotating frame

$$|\theta(t)\rangle = \mathrm{e}^{-\mathrm{i}\Omega t E_0}|\theta\rangle \tag{2.18}$$

where

$$\langle\theta(t)|E_0|\theta(t)\rangle = 0, \quad \langle\theta(t)|E|\theta(t)\rangle = \mathrm{e}^{-\mathrm{i}(\Omega t+\theta)}$$

It is easy to observe that the Hamiltonian (2.15) coincides with the average value of the following Hamiltonian over the phase states in the rotating frame,

$$H^F = \omega S_z + \Omega E_0 + g\left(E^\dagger S_- + E S_+\right) + g\left(E^\dagger S_+ + E S_-\right) \tag{2.19}$$

The Hamiltonian (2.19) is a Floquet form of the initial time-dependent Hamiltonian (2.15). Furthermore, we can, without loss of generality, set the global common phase θ equal to 0.

The Hamiltonian (2.19) is time independent. In the case, $g \ll \omega, \Omega$ it can be treated in a perturbative way using, for example, Lie-type perturbation theory (see Appendix 11.4). The general idea is the adiabatic elimination of all the terms that describe nonresonant transitions (for instance, antirotating terms like $S_- E + h.c.$) from Equation 2.19. This is done by applying in a perturbative way a series of small Lie-type transformations.

Let us choose the first transformation as follows:

$$U_1 = \mathrm{e}^{\varepsilon\left(E^\dagger S_+ - E S_-\right)}, \qquad \varepsilon = \frac{g}{\omega+\Omega} \ll 1$$

Taking into account only the terms up to the order ε^2 we have

$$\tilde{H} = U_1 H^F U_1^\dagger \approx H + \varepsilon[A, H] + \frac{\varepsilon^2}{2}[A,[A,H]] + \cdots$$

The value of the parameter ε is chosen in such a way that the antirotating term $g\left(E^\dagger S_+ + E S_-\right)$ disappears from the Hamiltonian (2.19). Indeed, by introducing the notations

$$H_0 = \omega S_z + \Omega E_0, \quad H_r = g\left(E^\dagger S_- + E S_+\right), \quad H_a = g\left(E^\dagger S_+ + E S_-\right)$$

we have

$$\begin{aligned}
U_1 H_0 U_1^\dagger &= H_0 - \varepsilon(\omega+\Omega)(E^\dagger S_+ + E S_-) - \varepsilon^2 2(\omega+\Omega) S_z E^\dagger E \\
U_1 H_r U_1^\dagger &= H_r + 2\varepsilon g(E^{\dagger 2} + E^2) S_z \\
&\quad - \varepsilon^2 g\left(E^{\dagger 3} S_+ + E^3 S_- + E^2 E^\dagger S_+ + E E^{\dagger 2} S_-\right) \\
U_1 H_a U_1^\dagger &= H_a + 4\varepsilon g E^\dagger E S_z - 2\varepsilon^2 g\left(E E^{\dagger 2} S_+ + E^\dagger E^2 S_-\right)
\end{aligned}$$

where the second term in the first equation cancels the antirotating term. Note also that the term

$$\frac{2g^2}{\omega+\Omega}\, S_z E^\dagger E$$

appears in the transformed Hamiltonian. Calculating the average over the states (2.18), $\langle\theta(t)|EE^\dagger|\theta(t)\rangle = 1$; we see that this term describes the effective shift of the atomic transition frequency, $2g^2/(\omega+\Omega)$, which is known as the *Bloch--Siegert shift*.

Let us now apply a second transformation to $\tilde{H}$:

$$U_2 = \mathrm{e}^{\delta\varepsilon(E^{\dagger 2}-E^2)S_z}, \qquad \delta = \frac{g}{\Omega} \ll 1$$

This removes the term $\varepsilon g 2(E^{\dagger\,2} + E^2)S_z$. Among the different terms that appear at this level in the transformed Hamiltonian, there is a term

$$-\frac{g^3}{\Omega(\Omega+\omega)}\,\left(E^3 S_+ + E^{\dagger\,3} S_-\right)$$

which describes the triple resonance and becomes important if the condition $\omega = 3\Omega$ is satisfied.

In this manner, we can remove all the nonresonant transitions from the Hamiltonian. However, we cannot remove the resonant terms. For instance, if we try to remove the principal term $g(S_+E + E^\dagger S_+)$ by applying a transformation $\exp\left[\tau\left(ES_+ - E^\dagger S_-\right)\right]$, then the value of the parameter $\tau = g/(\Omega-\omega)$ diverges as we approach the resonance, $\Omega \to \omega$. After application of these transformations, the terms in the transformed Hamiltonian can be classified into three groups:

1. *Diagonal terms*, which depend only on S_z and $E^\dagger E$, and whose average values represent the shifts of the system transition frequencies.
2. *Nondiagonal resonant terms*, which, under certain conditions imposed on the transition frequencies, describe resonant transitions; in other words, these terms become time independent in the rotating frame if certain relations between ω and Ω are fulfilled.
3. *Nondiagonal terms* that would never represent resonant interactions, and hence, can be eliminated.

This finally leads to an effective Hamiltonian that contains possible resonances in the system (see Chapter 9 and Appendix 11.4). Retaining only the leading orders of small parameters in the Hamiltonian, (i.e. neglecting terms of order $\delta^2\varepsilon^2$, we arrive at the following effective Hamiltonian:

$$H_{\mathrm{eff}}^{\mathrm{op}} = \omega S_z + \Omega E_0 + 2g\varepsilon E^\dagger E S_z + g\left(ES_+ + h.c.\right) - g\delta\varepsilon\left(E^3 S_+ + h.c.\right)$$

Averaging this Hamiltonian (8.31) over states (2.18) we obtain an effective time-dependent Hamiltonian

$$H_{\mathrm{eff}} = \Omega S_z + 2g\varepsilon S_z + g\left(\mathrm{e}^{-\mathrm{i}\Omega t}S_+ + h.c.\right) - g\delta\varepsilon\left(\mathrm{e}^{-3\mathrm{i}\Omega t}S_+ + h.c.\right)$$

It is shown in Chapter 9 that that only odd resonances, $\omega \approx \left(2k+1\right)\Omega$, appear in this system.

2.6
Collective Atomic Systems

Now let us consider the evolution of a collection of A two-level atoms in a classical external field. The Hamiltonian of the system has the form

$$H = \omega_0 S_z + \epsilon_1(t) S_x + \epsilon_2(t) S_y \tag{2.20}$$

where $S_{x,y,z}$ are collective operators (1.10) that satisfy the commutation relations of the $su(2)$ algebra (1.13); $\epsilon_1(t) = 2\mathrm{d}E_x(t)$ and $\epsilon_2(t) = 2\mathrm{d}E_y(t)$. In terms of the raising and lowering operators, $S_\pm = S_x \pm \mathrm{i}S_y$ the Hamiltonian (2.20) can be rewritten as

$$H = \omega_0 S_z + \epsilon_+ S_+ + \epsilon_- S_-, \quad \epsilon_+ = \frac{\epsilon_1 - \mathrm{i}\epsilon_2}{2} = \epsilon_-^*$$

In general, the parameters ϵ_+, ϵ_- may depend on time.

The evolution operator is the solution of the equation

$$\mathrm{i}\partial_t U = HU, \quad U(0) = I \tag{2.21}$$

We use a disentangled normal form (*Gauss decomposition*) for the operator U (see Appendix 11.3):

$$U(t) = \mathrm{e}^{\alpha(t)S_+}\,\mathrm{e}^{\beta(t)S_z}\,\mathrm{e}^{\gamma(t)S_-} \tag{2.22}$$

By substituting the evolution operator in the form (2.22) into Equation 2.21 we obtain

$$\begin{aligned}\mathrm{i}\left(\dot\alpha S_+ U + \dot\beta \mathrm{e}^{\alpha S_+} S_z \mathrm{e}^{-\alpha S_+} U + \dot\gamma\, \mathrm{e}^{\alpha S_+}\,\mathrm{e}^{\beta S_z} S_- \mathrm{e}^{-\beta S_z}\,\mathrm{e}^{-\alpha S_+} U\right)\\ = (\omega_0 S_z + \epsilon_+ S_+ + \epsilon_- S_-)\, U\end{aligned}$$

Using the following relations

$$\begin{aligned}\mathrm{e}^{\alpha S_+} S_z\, \mathrm{e}^{-\alpha S_+} = S_z - \alpha S_+, \quad \mathrm{e}^{\beta S_z} S_-\, \mathrm{e}^{-\beta S_z} = \mathrm{e}^{-\beta} S_-\\ \mathrm{e}^{\alpha S_+} S_-\, \mathrm{e}^{-\alpha S_+} = S_- + 2\alpha S_z - \alpha^2 S_+\end{aligned}$$

we obtain

$$\mathrm{i}\left(\dot\alpha S_+ + \dot\beta\,(S_z - \alpha S_+) + \dot\gamma\,\mathrm{e}^{-\beta}\left(S_- + 2\alpha S_z - \alpha^2 S_+\right)\right) = \omega_0 S_z + \epsilon_+ S_+ + \epsilon_- S_-$$

The operators $S_{\pm,z}$ are linearly independent. This leads us to the system of ordinary differential equations

$$\mathrm{i}\left(\dot\alpha - \dot\beta\alpha - \dot\gamma \mathrm{e}^{-\beta}\alpha^2\right) = \epsilon_+, \quad \mathrm{i}\left(\dot\beta + 2\dot\gamma\,\mathrm{e}^{-\beta}\alpha\right) = \omega_0, \quad \mathrm{i}\dot\gamma\,\mathrm{e}^{-\beta} = \epsilon_- \tag{2.23}$$

with the initial conditions $\alpha(0) = \beta(0) = \gamma(0) = 0$. Combining the second and third equations produces

$$\dot\beta = -\mathrm{i}\,(\omega_0 - 2\alpha\epsilon_-)$$

We can then insert this and the third of Equations 2.23 in the first of Equations 2.23 to obtain a Riccati-type equation for the coefficient alpha:

$$\mathrm{i}\dot{\alpha} = \omega_0\alpha + \epsilon_+ - \alpha^2\epsilon_-$$

The ansatz $\alpha = a\mathrm{e}^{-\mathrm{i}\omega_0 t}$ produces a differential equation for the variable a:

$$\mathrm{i}\dot{a} = \vartheta^* - \vartheta a^2, \quad a\,(0) = 0 \tag{2.24}$$

where $\vartheta = \epsilon_-\mathrm{e}^{-\mathrm{i}\omega_0 t}$.

Another substitution, $a = \mathrm{i}\dot{\psi}/\psi\vartheta$, reduces the Riccati Equation 2.24 to a second-order linear differential equation for the function ψ

$$-\ddot{\psi} + \dot{\psi}\frac{\dot{\vartheta}}{\vartheta} = |\vartheta|^2\psi$$

The substitution $\psi = \sqrt{\vartheta\,(t)}\xi$ eliminates the term with the first derivative and the equation for ξ becomes

$$\ddot{\xi} + \Omega^2(t)\xi = 0 \tag{2.25}$$

with the initial condition

$$\xi\,(0) = 1, \quad \dot{\xi}\,(0) = -\frac{\dot{\vartheta}\,(0)}{2\vartheta\,(0)}$$

The frequency $\Omega^2(t)$ is given by

$$\Omega^2(t) = |\vartheta|^2 + \frac{\ddot{\vartheta}}{2\vartheta} - \frac{3}{4}\left(\frac{\dot{\vartheta}}{\vartheta}\right)^2$$

and, in general, may be complex.

Let us consider the particular case of an evolution in a circularly polarized field, where, $\epsilon_1 = E\cos\omega t$ and $\epsilon_2 = E\sin\omega t$. Then,

$$\epsilon_- = \frac{E}{2}\mathrm{e}^{\mathrm{i}\omega t}, \quad \vartheta = \frac{E}{2}\mathrm{e}^{\mathrm{i}(\omega-\omega_0)t}, \quad \xi(0) = 1, \quad \dot{\xi}(0) = -\mathrm{i}\frac{\omega-\omega_0}{2}$$

and the frequency Ω^2 becomes time independent:

$$\Omega^2 = \frac{E^2}{4} + \frac{(\omega-\omega_0)^2}{4}$$

The equation for $\xi(t)$ is easily solved and yields

$$\xi = \cos\Omega t - \mathrm{i}\frac{\omega-\omega_0}{2\Omega}\sin\Omega t, \quad \psi = \sqrt{\frac{E}{2}}\,\mathrm{e}^{\mathrm{i}t(\omega-\omega_0)/2}\xi$$

For the coefficient a we have

$$a = -\frac{\mathrm{i}E\sin\Omega t\,\mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t}}{2\Omega\cos\Omega t - \mathrm{i}\,(\omega-\omega_0)\sin\Omega t}$$

At resonance, where $\omega = \omega_0$, we have

$$a = -\mathrm{i}\tan \Omega t, \quad \Omega = \frac{E}{2}$$

Now, we can find the coefficients of the Gauss decomposition:

$$\alpha = a\mathrm{e}^{-\mathrm{i}\omega_0 t} = -\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega t}\tan \Omega t$$

Since $\alpha\epsilon_- = -\mathrm{i}\Omega \tan \Omega t$, we have

$$\beta = -\mathrm{i}\omega_0 \int_0^t \mathrm{d}\tau + 2\mathrm{i}\int_0^t \alpha(\tau)\,\epsilon_-(\tau)\,\mathrm{d}\tau$$

$$= -\mathrm{i}\omega_0 t + 2\Omega \int_0^t \tan \Omega\tau \mathrm{d}\tau\beta = -\mathrm{i}\omega_0 t - 2\ln|\cos \Omega t|$$

From the equation for coefficient γ, we find

$$\dot{\gamma} = -\mathrm{i}\mathrm{e}^{\beta}\epsilon_- = -\mathrm{i}\Omega\cos^{-2}\Omega t$$

or, finally,

$$\gamma = -\mathrm{i}\tan \Omega t$$

The normal form (2.22) of the evolution operator allows us to calculate the probabilities of different atomic transitions. Let us suppose that the initial state is $|0, A\rangle$ (henceforth $|0\rangle$), that is, all the atoms are in their ground states. We calculate the probability $P_{00}(t)$ of their remaining in the nonexcited state after a time t,

$$P_{00}(t) = |\langle 0|U(t)|0\rangle|^2 = |\langle 0|\mathrm{e}^{\alpha S_+}\mathrm{e}^{\beta S_z}\mathrm{e}^{\gamma S_-}|0\rangle|^2 = |\langle 0|\mathrm{e}^{\beta S_z}|0\rangle|^2$$

Since $S_z|0\rangle = -A/2|0\rangle$, we obtain

$$P_{00}(t) = \mathrm{e}^{-A\mathrm{Re}\,\beta} = \cos^{2A}\Omega t$$

Let us evaluate the probability $\overline{P_{00}(t)}$ averaged over time $T \gg 2\pi/\Omega$

$$\overline{P_{00}(t)} = \frac{C_{2A}^{A}}{2^{2A}} = \frac{(2A)!}{2^{2A}\left(A!\right)^2}$$

where C_{2A}^{A} are the binomial coefficients. Thus, for one atom, we have $\overline{P_{00}(t)} = 1/2$ and, for two atoms, we have $\overline{P_{00}(t)} = 3/8$. In the case of a large number of atoms $A \gg 1$, using the Stirling formula $n! \sim \sqrt{2n\pi}\,n^n\mathrm{e}^{-n}$, we obtain

$$\overline{P_{00}(t)} \sim \frac{1}{\sqrt{A\pi}} \longrightarrow 0$$

that is, the system with a large number of atoms has a greater probability of becoming excited.

Next, we calculate the probability $P_{01}(t)$ of a transition from the ground state to the first excited state.

$$P_{01}(t) = |\langle 1|U(t)|0\rangle|^2 = |\langle 1|\mathrm{e}^{\alpha S_+}\mathrm{e}^{\beta S_z}\mathrm{e}^{\gamma S_-}|0\rangle|^2 = \mathrm{e}^{-A\,\mathrm{Re}\,\beta}|\langle 1|\mathrm{e}^{\alpha S_+}|0\rangle|^2$$

Considering that

$$|\langle 1|\mathrm{e}^{\alpha S_+}|0\rangle|^2 = |\alpha|^2|\langle 1|S_+|0\rangle|^2 = A\tan^2\Omega t$$

we obtain

$$P_{01}(t) = A\cos^{2A-2}\Omega t\sin^2\Omega t$$

In general, the probability $P_{0k}(t)$ to excite the state $|k, A\rangle$ (describing k excited atoms) can be calculated

$$P_{0k}(t) = |\langle k|U(t)|0\rangle|^2 = |\langle k|\mathrm{e}^{\alpha S_+}\mathrm{e}^{\beta S_z}\mathrm{e}^{\gamma S_-}|0\rangle|^2 = \mathrm{e}^{-A\,\mathrm{Re}\beta}|\langle k|\mathrm{e}^{\alpha S_+}|0\rangle|^2$$

Remembering that

$$S_+^k|0, A\rangle = \sqrt{\frac{A!k!}{(A-k)!}}|k, A\rangle$$

we obtain

$$|\langle k|\mathrm{e}^{\alpha S_+}|0\rangle|^2 = \frac{|\alpha|^{2k}}{(k!)^2}|\langle k|S_+^k|0\rangle|^2 = \frac{A!}{k!\,(A-k)!}\tan^{2k}\Omega t$$

This yields the probability $P_{0k}(t)$:

$$P_{0k}(t) = \frac{A!}{k!\,(A-k)!}\cos^{2A-2k}\Omega t\sin^{2k}\Omega t$$

It can easily be seen that the normalization condition holds ($\sum_{k=0}^{A} P_{0k}(t) = 1$), that is, the total probability is conserved (as it should be, for a unitary process). It is interesting to note that the probability of exciting all the atoms $P_{0A}(t)$ is equal to

$$P_{0A}(t) = \sin^{2A}\Omega t$$

Also, when averaged over many oscillations, it tends to zero for a large number of atoms:

$$\overline{P_{A0}(t)} = \frac{C_{2A}^{A}}{2A} \sim \frac{1}{\sqrt{A\pi}} \longrightarrow 0$$

Using the evolution operator in its normal form, we can determine the evolution of various physical quantities. We find, for example, the evolution of the atomic inversion (the difference between the number of excited and nonexcited atoms) starting with the nonexcited initial state $|0\rangle$:

$$\langle S_z(t)\rangle = \langle\psi(t)|S_z|\psi(t)\rangle = \langle 0|U^\dagger(t)\,S_z U(t)|0\rangle$$

Substituting the evolution operator in the normal form

$$U^{\dagger}(t) = \mathrm{e}^{\gamma^* S_+} \mathrm{e}^{\beta^* S_z} \mathrm{e}^{\alpha^* S_-}$$

we obtain

$$\langle S_z \rangle = \exp\left[-\frac{A}{2} \operatorname{Re} \beta \right] \langle 0 | \mathrm{e}^{\alpha^* S_-} S_z \mathrm{e}^{\alpha S_+} | 0 \rangle$$

where $\exp(-A/2 \operatorname{Re} \beta) = \cos^{2A} \Omega t$. The matrix element is calculated as in Section 1.3:

$$\Lambda = \langle 0 | \mathrm{e}^{\alpha^* S_-} S_z \mathrm{e}^{\alpha S_+} | 0 \rangle = \sum_{k=0}^{A} \frac{|\alpha|^{2k}}{(k!)^2} \langle 0 | S_-^k S_z S_+^k | 0 \rangle$$

Substituting the expression for $S_+^k |0\rangle$ in the equation above, we obtain

$$\Lambda = \sum_{k=0}^{A} \frac{|\alpha|^{2k}}{(k!)^2} \frac{A! k!}{(A-k)!} \left(k - \frac{A}{2} \right) = \left[|\alpha|^2 \frac{\partial}{\partial |\alpha|^2} - \frac{A}{2} \right] \left(1 + |\alpha|^2 \right)^A$$

If we recall that $|\alpha|^2 = \tan^2 \Omega t$, we obtain the atomic inversion as a function of time:

$$\langle S_z(t) \rangle = -\frac{A}{2} \cos 2\Omega t$$

However, the normal form of the evolution operator (2.22) is not adequate for determining the matrix element $\langle m | U(t) | k \rangle$, which gives us the probability of a transition from k to m excited atoms. To this end, we find another representation for the evolution operator $U(t)$. We look for the operator $U(t)$ in the following disentangled form (*Euler parameterization*):

$$U(t) = \mathrm{e}^{-\mathrm{i}\varphi S_z} \mathrm{e}^{-\mathrm{i}\theta S_y} \mathrm{e}^{-\mathrm{i}\psi S_z}$$

where (φ, θ, ψ) are the Euler angles, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$, $-2\pi \le \psi \le 2\pi$. There is a correspondence between parameters (φ, θ, ψ) and (α, β, γ). Since this relation does not depend on the representation, (i.e. on the number of atoms A), we find that $A = 1$ in the fundamental (two-dimensional) representation, which corresponds to a single atom). In the fundamental representation, the operators S_j, $j = x, y, z$ have the form

$$S_z = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S_x = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad S_y = \frac{1}{2} \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}$$

so that the evolution operator in the Euler parameterization is of the form

$$U(\varphi, \theta, \psi) = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\varphi/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\varphi/2} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\psi/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\psi/2} \end{bmatrix}$$

or

$$U(\psi,\varphi,\theta) = \begin{bmatrix} e^{-i(\varphi+\psi)/2}\cos\dfrac{\theta}{2} & -e^{-i(\varphi-\psi)/2}\sin\dfrac{\theta}{2} \\ e^{i(\varphi-\psi)/2}\sin\dfrac{\theta}{2} & e^{i(\varphi+\psi)/2}\cos\dfrac{\theta}{2} \end{bmatrix}$$

Likewise, in the fundamental representation the operators $S_\pm$ have the form

$$S_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad S_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

which yields the evolution operator in the disentangled Gauss form

$$U(\alpha,\beta,\gamma) = e^{\alpha S_+} e^{\beta S_z} e^{\gamma S_-}$$

in the fundamental representation

$$U(\alpha,\beta,\gamma) = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^{\beta/2} & 0 \\ 0 & e^{-\beta/2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}$$

or, finally,

$$U(\alpha,\beta,\gamma) = \begin{bmatrix} e^{\beta} + \alpha\gamma & \alpha \\ \gamma & 1 \end{bmatrix} e^{-\beta/2}$$

Equating the evolution operator in both parameterizations,

$$U(\varphi,\theta,\psi) = U(\alpha,\beta,\gamma)$$

we obtain the relations between the two sets of parameters:

$$\alpha = -\tan\frac{\theta}{2}\,e^{-i\varphi}, \quad \gamma = \tan\frac{\theta}{2}\,e^{-i\psi}, \quad \beta = -2\ln\cos\frac{\theta}{2} - i(\varphi+\psi)$$

or, introducing, $\alpha = |\alpha|\exp(i\arg\alpha)$, $\gamma = |\gamma|\exp(i\arg\gamma)$, we obtain

$$|\alpha| = |\gamma| = \tan\frac{\theta}{2}, \quad \varphi = \pi - \arg\alpha, \quad \psi = -\arg\gamma$$

We take note of a useful relation between the parameters α and β that follows from the unitarity of U:

$$\operatorname{Re}\beta = -2\ln\frac{1}{\sqrt{1+|\alpha|^2}} = \ln\left(1+|\alpha|^2\right)$$

For example, in the case of the interaction with the circularly polarized field, it is easy to obtain the parameters (φ,θ,ψ):

$$|\alpha| = \tan\Omega t, \quad \psi = -\frac{\pi}{2} + \omega_0 t, \quad \varphi = \frac{\pi}{2} \tag{2.26}$$

With the evolution operator $U(t)$ in the Euler parameterization, the transition amplitude between two arbitrary Dicke states $|k,A\rangle$ and $|m,A\rangle$ is reduced to a standard expression that is well known in group theory:

$$\langle m, A|\, U(t)|k, A\rangle = \left\langle m, A|\mathrm{e}^{-\mathrm{i}\varphi S_z}\mathrm{e}^{-\mathrm{i}\theta S_y}\mathrm{e}^{-\mathrm{i}\psi S_z}|k, A\right\rangle$$

Since $S_z|k, A\rangle = \left(k - A/2\right)|k, A\rangle$, we have

$$\langle m, A|\, U(t)|k, A\rangle = \mathrm{e}^{-\mathrm{i}\varphi(m-A/2)}\mathrm{e}^{-\mathrm{i}\psi(k-A/2)}d^A_{mk}(\theta)$$

where

$$d^A_{mk}(\theta) = \langle m, A|\mathrm{e}^{-\mathrm{i}\theta S_y}|k, A\rangle$$

is the Wigner d function (see Appendix 11.5). For example, in the case of a single atom, $A = 1$, in a circularly polarized resonant field (2.26), the evolution operator takes the form

$$U(t) = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\omega t/2}\cos\Omega t & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\omega t/2}\sin\Omega t \\ -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\omega t/2}\sin\Omega t & \mathrm{e}^{\mathrm{i}\omega t/2}\cos\Omega t \end{bmatrix}$$

The probability of a transition from state k to state m (from k to m excited atoms) is equal to

$$P_{mk}(t) = |d^A_{mk}(\theta)|^2$$

2.7
Atomic System in a Field of a Single Pulse

The Gauss decomposition for the evolution operator (2.22) discussed in the previous section is not useful if the external field is a pulse with vanishing amplitude at $t \to \pm\infty$. In particular, the reduction of the system of Equations 2.23 to a single one Equation 2.25 is impossible since the coefficients of this equation diverge when $t \to \pm\infty$. Thus, when the field is a pulse, the evolution operator should be represented in a form different from Equation 2.22.

Let us suppose that the interaction Hamiltonian has the form

$$H = \omega_0 S_z + f(t)S_x \tag{2.27}$$

where $f(t)$ must be real if H is to be Hermitian. We are interested in the form of the evolution $U(t)$ generated by Equation 2.27 as $t \to \infty$.

First, we represent the evolution operator as

$$U_{\mathrm{at}}(t) = \exp\left(-\mathrm{i}\frac{\pi}{2}S_x\right)\tilde{U}(t)$$

The operator $\tilde{U}(t)$ is a solution of the equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{U} = \tilde{H}\tilde{U} \tag{2.28}$$

where the Hamiltonian $\tilde{H}$ takes on the form

$$\tilde{H} = \exp\left(\mathrm{i}\frac{\pi}{2}S_x\right)H\exp\left(-\mathrm{i}\frac{\pi}{2}S_x\right) = \omega_0 S_y + f(t)S_x$$

or, in terms of the raising and lowering operators,

$$\tilde{H} = \epsilon_+ S_+ + \epsilon_- S_-, \quad \epsilon_+ = \frac{1}{2}\left(f(t) - \mathrm{i}\omega_0\right) = \epsilon_-^* \tag{2.29}$$

It can easily be seen that the functions $\epsilon_\pm$ cannot approach zero since $f(t)$ is real.

We can now use the Gauss decomposition (2.22) for the operator $\tilde{U}$, but with different initial conditions for the functions α, β, and γ. We suppose there is no interaction at time $t = -\infty$: $f(t \to -\infty) = 0$, so that, at $t = -\infty$ the evolution operator has an asymptotic form corresponding to the free evolution

$$U_{\mathrm{at}}(t \to -\infty) = \exp\left(-\mathrm{i}\frac{\pi}{2}S_x\right)\tilde{U}(t \to -\infty) = \exp\left(-\mathrm{i}\omega_0 t S_z\right)$$

This determines the initial conditions of parameters α, β, and γ:

$$\alpha(-\infty) = \mathrm{i}, \quad \mathrm{e}^{\beta(-\infty)} = \frac{1}{2}\mathrm{e}^{-\mathrm{i}\omega_0 t}, \quad \gamma(-\infty) = \mathrm{i}\mathrm{e}^{-\mathrm{i}\omega_0 t} \tag{2.30}$$

The functions α, β, and γ satisfy the system of equations analogous to Equation 2.23:

$$\dot{\beta} = 2\mathrm{i}\alpha\epsilon_-, \quad \mathrm{i}\dot{\alpha} = \epsilon_+ - \alpha^2\epsilon_-, \quad \mathrm{i}\dot{\gamma}\mathrm{e}^{-\beta} = \epsilon_- \tag{2.31}$$

where the functions $\epsilon_\pm$ are defined in Equation 2.29. The substitutions

$$\alpha = \mathrm{i}\frac{\dot{\eta}}{\eta\epsilon_-}, \quad \eta = \sqrt{\frac{\epsilon_-(t)}{\epsilon_-(-\infty)}}\xi$$

lead to the following equation for ξ:

$$\ddot{\xi} + \Omega^2(t)\,\xi = 0 \tag{2.32}$$

where

$$\Omega^2(t) = \frac{\ddot{f}}{2(f + \mathrm{i}\omega_0)} - \frac{3}{4}\left(\frac{\dot{f}}{f + \mathrm{i}\omega_0}\right)^2 + \frac{1}{4}\left(f^2 + \omega_0^2\right) \tag{2.33}$$

Using $\epsilon_-(-\infty) = \mathrm{i}\omega_0/2$ we obtain $\eta(-\infty)$ and $\xi(-\infty)$

$$\eta(-\infty) = \xi(-\infty) = \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\omega_0 t}$$

Also, since the interaction disappears for $t \to \infty$, we have $\epsilon_-(\infty) = \mathrm{i}\omega_0/2$ and thus $\Omega^2(t \to \infty) = \omega_0^2/4$. This means that for $t \to \infty$, the solution of Equation (2.32) takes the following form:

$$\eta(t \to \infty) = \xi(t \to \infty) = \frac{1}{\sqrt{2}}\left(\mu\mathrm{e}^{\mathrm{i}\omega_0 t} + \nu\mathrm{e}^{-\mathrm{i}\omega_0 t}\right) \tag{2.34}$$

where the coefficients μ and ν are determined from the asymptotic of the solution of Equation 2.32 with $\Omega^2(t)$ defined in Equation 2.33. From Equation 2.34, we obtain $\alpha(t \to \infty)$:

$$\alpha(t \to \infty) = \mathrm{i}\frac{\mu\mathrm{e}^{\mathrm{i}\omega_0 t} - \nu\mathrm{e}^{-\mathrm{i}\omega_0 t}}{\mu\mathrm{e}^{\mathrm{i}\omega_0 t} + \nu\mathrm{e}^{-\mathrm{i}\omega_0 t}}$$

From the first equation of the system (2.31), we find $\dot{\beta}$:

$$\dot{\beta} = i2\alpha\epsilon_- = -2\frac{\dot{\eta}}{\eta}$$

Integrating this last equation, we obtain

$$\beta = -2\ln\eta \tag{2.35}$$

which satisfies the initial conditions

$$e^{\beta(-\infty)} = \eta^2(-\infty) = \frac{1}{2}e^{-i\omega_0 t}$$

Thus, we have

$$\beta(t \to \infty) = -2\ln\xi(t \to \infty)$$

It is easy to find the relation between parameters α, β, and γ:

$$\gamma = -\frac{\alpha^*}{1+|\alpha|^2}e^{\beta} \tag{2.36}$$

From this, we obtain

$$\gamma(t \to \infty) = -\frac{\alpha^*(t \to \infty)}{1+|\alpha(t \to \infty)|^2\xi^2(t \to \infty)}$$

Considering the relation,

$$\operatorname{Re}\beta = \ln\left(1+|\alpha|^2\right) \tag{2.37}$$

which implies $|\alpha| = |\gamma|$, it is clear that the asymptotic parameters μ and ν in Equation 2.34 satisfy the relation

$$|\mu|^2 + |\nu|^2 = 1 \tag{2.38}$$

and have the meaning of the transition and reflection coefficients in the potential given by Equation 2.33. Finally, the asymptotic form ($t \to \infty$) of the evolution operator is

$$U(t \to \infty) = e^{-i(\pi/2)S_x}e^{\alpha_\infty S_+}e^{\beta_\infty S_z}e^{\gamma_\infty S_-} \tag{2.39}$$

We now assume that initially (for $t \to -\infty$), the atomic system was in its ground state $|0, A\rangle$ and determine the probability that k atoms get excited after the pulse has ended, that is, for $t \to \infty$. From Equation 2.39 we obtain the matrix element

$$\langle k|U(t \to \infty)|0\rangle = e^{-\beta_\infty A/2}\langle k|e^{-i(\pi/2)S_x}e^{\alpha_\infty S_+}|0\rangle$$

Remembering that

$$\langle m|e^{\alpha_\infty S_+}|0\rangle = \alpha_\infty^m\sqrt{\frac{A!}{(A-m)!m!}}$$

we obtain

$$\langle k|\,U(\infty)|0\rangle = \mathrm{e}^{-\beta_\infty A/2} \sum_{m=0}^{A} P_{km}\left(-\frac{\pi}{2}\right)\,\alpha_\infty^m \sqrt{\frac{A!}{m!\,(A-m)!}}$$

where

$$P_{km}(\theta) = \langle k|\mathrm{e}^{\mathrm{i}\theta S_x}|m\rangle$$

(see Appendix 11.5). Applying the summation rule

$$\sum_{m=0}^{A} P_{km}(\theta)\,\frac{\alpha^m}{\sqrt{m!\,(A-m)!}} = \frac{1}{\sqrt{k!\,(A-k)!}}\,\frac{(\alpha\cos\theta/2 + \mathrm{i}\sin\theta/2)^k}{(\cos\theta/2 + \mathrm{i}\alpha\sin\theta/2)^{k-A}}$$

and using the relation (2.37), we find

$$|\langle k|\,U(\infty)|0\rangle|^2 = \frac{A!}{k!\,(A-k)!}\,\frac{|\alpha_\infty + \mathrm{i}|^{2A}}{\left(|\alpha_\infty|^2+1\right)^A}\left|\frac{\alpha_\infty - \mathrm{i}}{\alpha_\infty + \mathrm{i}}\right|^{2k}$$

It is easy to verify that the normalization is satisfied:

$$\sum_{k=0}^{A} |\langle k|\,U(\infty)|0\rangle|^2 = 1$$

Representing α_∞ as a function of the parameters μ and ν and considering the relation (2.38), we obtain

$$|\langle k|\,U(\infty)|0\rangle|^2 = \frac{A!}{k!\,(A-k)!}|\mu|^{2(A-k)}\left(1-|\mu|^2\right)^k \tag{2.40}$$

Thus, the probability of excitation is expressed in terms of the reflection coefficient μ. The expression (2.40) completely solves the problem of the excitation of a set of two-level atoms by a classical pulse.

In the adiabatic case, when the derivatives $\dot{f}$ and $\ddot{f}$ are negligible, we have

$$\Omega^2(t) \approx \frac{1}{4}\left(f^2(t) + \omega_0^2\right) \tag{2.41}$$

and the *WKB* solution for Equation (2.32) has the form

$$\xi(t) = \frac{1}{\sqrt{\Omega}}\mathrm{e}^{\pm\mathrm{i}\int_{-\infty}^{t}\Omega(\tau)\,\mathrm{d}\tau}$$

In this case, the parameter $|\mu|^2$ is given by the equation

$$|\mu|^2 = \left|\frac{1}{4}\int_{-\infty}^{\infty}\frac{\mathrm{d}\tau}{\sqrt{\Omega\,(\tau)}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\Omega/\mathrm{d}\tau}{\Omega^{3/2}}\right)\exp\left[-2\mathrm{i}\int_{-\infty}^{\infty}\Omega(z)\,\mathrm{d}z\right]\right|^2$$

which is valid if $|\mu|^2 \ll 1$.

2.8
Problems

2.1 Describe the evolution of A atoms initially in the coherent state $|\xi = 1\rangle$ under the influence of the Hamiltonian

$$H = \chi S_z^2$$

Find the evolution of the Bloch vector.

2.2 The Hamiltonian for one atom in the constant field has the form

$$H = \frac{1}{2}\omega_0\sigma_z + \frac{1}{2}\omega_1\sigma_x$$

Find the equations for the parameters (θ, φ) that determine the position of the state on the Bloch sphere. Solve these equations in the case when $\omega_0 \gg \omega_1$. Analyze the geometrical representation of this evolution.

2.3 For the case of $A = 2$ atoms, find the evolution of the Bloch vector in the circularly polarized field if initially one of the atoms was excited.

2.4 Find the evolution operator of a collection of A two-level atoms for the Hamiltonian

$$H = \omega S_z + \delta_T(t)\, S_x, \quad \delta_T(t) = \chi \sum_{n=0}^{\infty} \delta(t - nT)$$

at the instant $NT + \varepsilon$, $\varepsilon \to +0$ and analyze the evolution of the initial nonexcited atomic state.

Hint: Represent the evolution operator during a single period as

$$U(T) = \mathrm{e}^{-\mathrm{i}\omega T S_z}\mathrm{e}^{-\mathrm{i}\chi S_x} = \mathrm{e}^{-\mathrm{i}(\alpha S_z + \xi S_+ + \xi^* S_-)}$$

2.5 Find the evolution operator for a three-level atom with Ξ configuration whose dynamics is described by the Hamiltonian ($E_1 \le E_2 \le E_3$)

$$H = E_1 S^{11} + E_2 S^{22} + E_3 S^{33} + g_{13}(\mathrm{e}^{-\mathrm{i}\omega_1 t} S_+^{12} + \mathrm{e}^{\mathrm{i}\omega_1 t} S_-^{12}) + g_{23}(\mathrm{e}^{-\mathrm{i}\omega_2 t} S_+^{23} + \mathrm{e}^{\mathrm{i}\omega_2 t} S_-^{23})$$

Analyze the case: $E_2 - E_1 = E_3 - E_2$.

2.6 Find the evolution operator for a three-level atom with Λ configuration whose dynamics is described by the Hamiltonian ($E_1 \le E_2 \le E_3$)

$$H = E_1 S^{11} + E_2 S^{22} + E_3 S^{33} + g_{13}(\mathrm{e}^{-\mathrm{i}\omega_1 t} S_+^{13} + \mathrm{e}^{\mathrm{i}\omega_1 t} S_-^{13}) + g_{23}(\mathrm{e}^{-\mathrm{i}\omega_2 t} S_+^{23} + \mathrm{e}^{\mathrm{i}\omega_2 t} S_-^{23})$$

Analyze the case: $E_1 = E_2$, $\omega_1 = \omega_2 \gg |E_3 - E_2|$.

3
Quantized Electromagnetic Field

3.1
Quantization of the Electromagnetic Field

Maxwell's equations for an electromagnetic field in vacuum have the form ($c = 1$)

$$\nabla \times \boldsymbol{\mathcal{E}} = -\mathbf{B}, \quad \nabla \cdot \boldsymbol{\mathcal{E}} = 0 \tag{3.1}$$
$$\nabla \times \mathbf{H} = \mathbf{E}, \qquad \nabla \cdot \mathbf{B} = 0$$

In the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0,\ \varphi = 0$, the field vectors are determined as

$$\boldsymbol{\mathcal{E}} = -\partial_t \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A} \tag{3.2}$$

and the potential vector $\mathbf{A}$ satisfies the wave equation

$$\Delta \mathbf{A} - \partial_t^2 \mathbf{A} = 0 \tag{3.3}$$

We suppose that the field is within a cubic cavity with a volume $V = L \times L \times L$; we also assume periodic boundary conditions. Thus, the solution of the wave equation can be represented as a series

$$\mathbf{A} = \sqrt{\frac{2\pi}{V}} \sum_{\mathbf{k}} \left(\boldsymbol{\epsilon}_{\mathbf{k}} A_{\mathbf{k}} \mathrm{e}^{-\mathrm{i}(\omega_k t - \mathbf{k}\,\mathbf{r})} + c.c. \right) \tag{3.4}$$

where $\mathbf{k}$ is the wave vector, $A_{\mathbf{k}}$ is the amplitude, and $\boldsymbol{\epsilon}_{\mathbf{k}} = \left(\epsilon_{\mathbf{k}}^1, \epsilon_{\mathbf{k}}^2\right)$ is the polarization vector orthogonal to $\mathbf{k}$, i.e. $\mathbf{k} \perp \boldsymbol{\epsilon}_k$, with discrete components, $k_{x,y,z} = 2\pi n_{x,y,z}/L$, where n_x, n_y, n_z are positive integers and the frequency of each mode is determined by the dispersion relation $\omega_k = \sqrt{k_x^2 + k_y^2 + k_z^2}$.

Substituting the expressions (3.2) for $\mathbf{B}$ and $\boldsymbol{\mathcal{E}}$ in terms of $\mathbf{A}$ in the free field Hamiltonian

$$H = \frac{1}{8\pi} \int \mathrm{d}V \left(\mathcal{E}^2 + B^2 \right) \tag{3.5}$$

we obtain

$$H = \sum_k \omega_k^2 A_k A_k^* \tag{3.6}$$

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(From now on, we omit the vector sign in the indices.) Introducing the variables Q_k and P_k as follows

$$A_k = \frac{1}{\sqrt{2}}\left(Q_k + \mathrm{i}\frac{P_k}{\omega_k}\right) \tag{3.7}$$

the Hamiltonian (3.6) becomes

$$H = \frac{1}{2}\sum_k (Q_k^2\omega_k^2 + P_k^2) \tag{3.8}$$

which corresponds to a system on decoupled oscillators.

The canonical quantization of the field consists in substituting for variables Q_k and P_k the position and momentum operators that satisfy the commutation relations

$$\left[Q_k, P_m\right] = \mathrm{i}\hbar\delta_{km}, \quad \left[Q_k, Q_m\right] = 0, \quad [P_k, P_m] = 0 \tag{3.9}$$

This means that the amplitudes A_k and A_m^* become noncommuting operators $\left[A_k, A_m^\dagger\right] = \delta_{km}\hbar/\omega_k$, so that rescaled operators

$$a_k = \sqrt{\frac{\omega_k}{\hbar}}A_k, \quad a_k^\dagger = \sqrt{\frac{\omega_k}{\hbar}}A_k^\dagger \tag{3.10}$$

satisfy the bosonic commutation relations

$$[a_m, a_k^\dagger] = \delta_{mk} \tag{3.11}$$

In terms of the operators $a_k, a_k^\dagger$ the field Hamiltonian takes the form

$$H = \sum_k \hbar\omega_k a_k^\dagger a_k + \frac{1}{2}\sum_k \hbar\omega_k \tag{3.12}$$

The first sum corresponds to an infinite set of noninteracting quantum oscillators, which allows us to use all the techniques developed in the study of the quantum harmonic oscillator. Each term in the second sum represent the vacuum energy of the appropriate mode and will be omitted in future calculations.

The total Hilbert space is a direct product of Hilbert spaces corresponding to each field mode. This total Hilbert space is spanned by vectors of the form

$$|n_1\rangle \otimes |n_2\rangle \otimes \cdots \otimes |n_k\rangle \cdots \tag{3.13}$$

where the vectors $|n_k\rangle$ describe the states with a defined energy, that is, with a fixed number of photons in the kth field mode. The states $|n_k\rangle$, usually called *number states* or *Fock states*, are constructed as eigenstates of the excitation-number operator in the kth field mode $\hat{n}_k = a_k^\dagger a_k$:

$$\hat{n}_k|n_k\rangle = n_k|n_k\rangle \tag{3.14}$$

and form an orthonormal basis, $\langle n_m|n_k\rangle = \delta_{mk}$, satisfying the following resolution of the identity:

$$\sum_{n_k=1}^{\infty} |n_k\rangle\langle n_k| = I \tag{3.15}$$

The operators a_k and $a_k^\dagger$ act in a standard way (see Appendix 11.1)

$$a_k|n_1\rangle|n_2\rangle\cdots|n_k\rangle|n_{k+1}\rangle\cdots = \sqrt{n_k}\,|n_1\rangle|n_2\rangle\cdots|n_k-1\rangle|n_{k+1}\rangle\cdots$$

and

$$a_k^\dagger|n_1\rangle|n_2\rangle\cdots|n_k\rangle|n_{k+1}\rangle\cdots = \sqrt{n_k+1}|n_1\rangle|n_2\rangle\cdots|n_k+1\rangle|n_{k+1}\rangle\cdots \tag{3.16}$$

Finally, the field operators in terms of $a_k^\dagger$ and a_k are

$$\mathrm{A} = \sum_k \boldsymbol{\epsilon}_k\sqrt{\frac{2\hbar\pi}{V\omega_k}}\left(a_k\mathrm{e}^{-\mathrm{i}(\omega_k t-\mathbf{k}\,\mathbf{r})} + a_k^\dagger\mathrm{e}^{\mathrm{i}(\omega_k t-\mathbf{k}\,\mathbf{r})}\right) \tag{3.17}$$

and

$$\mathcal{E} = \mathrm{i}\sum_k \boldsymbol{\epsilon}_k\sqrt{\frac{2\hbar\pi\omega_k}{V}}\left(a_k\mathrm{e}^{-\mathrm{i}(\omega_k t-\mathbf{k}\,\mathbf{r})} - a_k^\dagger\mathrm{e}^{\mathrm{i}(\omega_k t-\mathbf{k}\,\mathbf{r})}\right) \tag{3.18}$$

Henceforth, we work mainly with one only field mode, so that only one term from the corresponding sums is needed. In particular, the Hamiltonian for a free field with a single mode is

$$H = \hbar\,\omega\hat{n}, \quad \hat{n} = a^\dagger a \tag{3.19}$$

and the representation space coincides with the space of a one-dimensional harmonic oscillator, $\hat{n}|n\rangle = n|n\rangle$.

3.2
Coherent States

The coherent states for a field mode can be defined as (normalized) eigenstates of the operator a:

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad \langle\alpha|\alpha\rangle = 1 \tag{3.20}$$

where $\alpha = |\alpha|\ \mathrm{e}^{\mathrm{i}\varphi}$ is a complex number. Substituting the expansion of the state $|\alpha\rangle$ in the number state basis $|n\rangle$

$$|\alpha\rangle = \sum_n c_n|n\rangle \tag{3.21}$$

into Equation 3.20

$$\sum_n \sqrt{n}\,c_n|n-1\rangle = \alpha\sum_n c_n|n\rangle \tag{3.22}$$

we find the recursion relation for the coefficients c_n: $\sqrt{n+1}\, c_{n+1} = \alpha\, c_n$, which can be easily solved:

$$c_n = \frac{\alpha^n}{\sqrt{n!}}\, c_0 \tag{3.23}$$

where c_0 is determined from the normalization condition $\langle\alpha|\alpha\rangle = 1$. This yields $c_0 = \exp\left(-|\alpha|^2/2\right)$. Thus, we get

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{3.24}$$

which means that the coherent states can be considered as generating functions for the number states:

$$|n\rangle = \frac{1}{\sqrt{n!}} \frac{\mathrm{d}^n}{\mathrm{d}\alpha^n} \left(\mathrm{e}^{(|\alpha|^2/2)} |\alpha\rangle \right)_{\alpha=0} \tag{3.25}$$

Likewise, we obtain for any operator $\hat{X}$, the relation between its matrix elements in the basis of coherent states and in the basis of number states:

$$\langle m|\hat{X}|n\rangle = \frac{1}{\sqrt{m!}\sqrt{n!}} \frac{\mathrm{d}^n}{\mathrm{d}\beta^n} \frac{\mathrm{d}^n}{\mathrm{d}\alpha^{*n}} \left(\mathrm{e}^{(|\alpha|^2/2)+(|\beta|^2/2)} \langle\beta|\hat{X}|\alpha\rangle \right)_{\alpha,\beta=0} \tag{3.26}$$

Since $\langle\alpha|a|\alpha\rangle = \alpha$, we can find the expectation value of the electric field intensity in the coherent state (in a single mode case):

$$\begin{aligned} \langle\alpha|\boldsymbol{\mathcal{E}}|\alpha\rangle &= \mathrm{i}\sqrt{\frac{2\pi\hbar\omega}{V}} \left(\alpha \mathrm{e}^{-\mathrm{i}(\omega t - \mathbf{k}\mathbf{r})} - \alpha^* \mathrm{e}^{\mathrm{i}(\omega t - \mathbf{k}\mathbf{r})} \right) \\ &= 2\,|\alpha| \sqrt{\frac{2\pi\hbar\omega}{V}} \sin\left(\omega t - \mathbf{k}\mathbf{r} - \varphi\right) \end{aligned} \tag{3.27}$$

which has the form of the *classical* electric field intensity with $\mathcal{E}_0 = 2\,|\alpha|\sqrt{2\pi\hbar\omega/V}$. In the same manner, we find for $\langle\alpha|\mathcal{E}^2|\alpha\rangle$ and the fluctuation $\sigma_{\mathcal{E}}$

$$\langle\alpha|\mathcal{E}^2|\alpha\rangle = -\frac{2\pi\hbar\omega}{V} \left[\left(\alpha \mathrm{e}^{-\mathrm{i}(\omega t - \mathbf{k}\mathbf{r})} - \alpha^* \mathrm{e}^{\mathrm{i}(\omega t - \mathbf{k}\mathbf{r})} \right)^2 - 1 \right] \tag{3.28}$$

and

$$\sigma_{\mathcal{E}} = \langle\alpha|\mathcal{E}^2|\alpha\rangle - \langle\alpha|\mathcal{E}|\alpha\rangle^2 = \frac{2\pi\hbar\omega}{V} \tag{3.29}$$

Note that in the case of an infinite number of modes, the fluctuation $\sigma_{\mathcal{E}}$ would be proportional to $\hbar \sum_k \omega_k$, which is a divergent sum.

3.3
Properties of the Coherent States

The most important physical property of the coherent states consists in minimizing the Schrodinger–Robertson uncertainty relation. Using units $\omega = 1$ and $\hbar = 1$, we can introduce dimensionless position and momentum operators:

$$q = \frac{a + a^\dagger}{\sqrt{2}}, \quad p = -\frac{\mathrm{i}}{\sqrt{2}}\left(a - a^\dagger\right), \quad [q, p] = \mathrm{i} \tag{3.30}$$

The uncertainty relation has the form [26]

$$\sigma_q \sigma_p - \sigma_{pq}^2 \geq \frac{1}{4} \tag{3.31}$$

where

$$\sigma_q = \langle\alpha|q^2|\alpha\rangle - \langle\alpha|q|\alpha\rangle^2, \quad \sigma_p = \langle\alpha|p^2|\alpha\rangle - \langle\alpha|p|\alpha\rangle^2 \tag{3.32}$$

$$\sigma_{pq} = \frac{1}{2}\langle pq + qp\rangle - \langle p\rangle\langle q\rangle \tag{3.33}$$

Direct calculations give the following average values of canonical operators and their momenta in the coherent state $|\alpha\rangle$:

$$\langle\alpha|q|\alpha\rangle = \sqrt{2}\,\mathrm{Re}\,\alpha, \quad \langle\alpha|p|\alpha\rangle = \sqrt{2}\,\mathrm{Im}\,\alpha \tag{3.34}$$

$$\langle\alpha|q^2|\alpha\rangle = 2(\mathrm{Re}\,\alpha)^2 + \frac{1}{2}, \quad \langle\alpha|p^2|\alpha\rangle = 2(\mathrm{Im}\,\alpha)^2 + \frac{1}{2} \tag{3.35}$$

leading to

$$\sigma_q = \sigma_p = \frac{1}{2}, \quad \sigma_{pq} = 0 \tag{3.36}$$

This means that the canonical variables have the same dispersion and are not correlated in the coherent states. For the coherent states, the Schrodinger–Robertson uncertainty relation (3.31) is minimized:

$$\sigma_q \sigma_p = \frac{1}{4} \tag{3.37}$$

The photon distribution function

$$P_n = |\langle n|\alpha\rangle|^2, \quad \sum_n P_n = 1 \tag{3.38}$$

which determines the probability of detecting n photons when the field is in the coherent state $|\alpha\rangle$, $\alpha = |\alpha|\mathrm{e}^{\mathrm{i}\varphi}$, is a Poisson distribution

$$P_n = \mathrm{e}^{-|\alpha|^2} \frac{|\alpha|^2}{n!} \tag{3.39}$$

When we note that the average number of photons in the state $|\alpha\rangle$ is $\overline{n} = \langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2$, we can rewrite the above distribution as $P_n = \mathrm{e}^{-\overline{n}}\overline{n}^n/n!$. One can easily verify that the function

$$W(\xi) = \sum_n P_n \mathrm{e}^{\xi n} = \exp\left[\overline{n}(\mathrm{e}^{\xi} - 1)\right] \tag{3.40}$$

is a moment generating function for $\hat{n}^k$, since

$$\langle\alpha|\hat{n}^k|\alpha\rangle = \frac{\mathrm{d}^k}{\mathrm{d}\xi^k} W(\xi)|_{\xi=0} \tag{3.41}$$

From Equation 3.40 we obtain

$$\langle\alpha|\hat{n}^2|\alpha\rangle = \sum_n n^2 P_n = \frac{d^2}{d\xi^2} W(\xi)|_{\xi=0} = \overline{n}^2 + \overline{n} \tag{3.42}$$

thus, the photon number fluctuations is equal to the number of photons

$$\sigma_n = \left\langle \hat{n}^2 \right\rangle - \left\langle \hat{n} \right\rangle^2 = \overline{n} \tag{3.43}$$

The relative fluctuation is given by

$$\delta_n = \frac{\sqrt{\sigma_n}}{\overline{n}} = \frac{1}{\sqrt{\overline{n}}} \tag{3.44}$$

and in the limit of large average photon numbers, ($\overline{n} \gg 1$), the relative fluctuation tends to zero $\delta_n \to 0$. In this case, the distribution function P_n approaches a Gaussian distribution (by the central limit theorem):

$$P_n \longrightarrow \frac{1}{\sqrt{2\pi\overline{n}}} e^{-(n-\overline{n})^2/2\overline{n}} \tag{3.45}$$

The coherent states form a basis in the space of field states:

$$\frac{1}{\pi}\int d^2\alpha \, |\alpha\rangle\langle\alpha| = \frac{1}{\pi}\int d\mathrm{Re}\,\alpha \, d\mathrm{Im}\,\alpha \, |\alpha\rangle\langle\alpha| = 1 \tag{3.46}$$

where integration is done over the entire complex plane (which plays the role of the phase space for the field, see Appendix 11.2).

The coherent states are not orthogonal; their overlap is given by

$$\langle\alpha|\beta\rangle = \exp\left[-\frac{|\alpha|^2 + |\beta|^2}{2}\right] \sum_n \frac{(\alpha^*\beta)^n}{n!} = \exp\left[-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha^*\beta\right] \tag{3.47}$$

However, the transition amplitude $\langle\alpha|\beta\rangle$ is a Gaussian function, with transition probability given by

$$|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2} \tag{3.48}$$

therefore, they can be considered as approximately orthogonal for essentially different values of α and β. From Equation 3.47, it follows that the basis spanned by the coherent states is overcomplete, in the sense that

$$|\beta\rangle = \frac{1}{\pi}\int d^2\alpha \, |\alpha\rangle \, \langle\alpha|\beta\rangle = \frac{1}{\pi}\int d^2\alpha \, |\alpha\rangle \, e^{-|\alpha|^2/2-|\beta|^2/2+\alpha^*\beta} \tag{3.49}$$

There is an interesting connection between field coherent states and coherent states of an atomic system (1.17) consisting of A indistinguishable two-level atoms. Indeed, let us consider two-mode field coherent state

$$|\alpha_1\rangle \, |\alpha_2\rangle = e^{-(|\alpha_1|^2+|\alpha_2|^2/2)} \sum_{n,m=0}^{\infty} \frac{\alpha_1^n \, \alpha_2^m}{\sqrt{n!m!}} \, |n\rangle \, |m\rangle \tag{3.50}$$

If we now introduce the new variables $A = n + m$ and $k = (n - m)/2 + A/2$

$$|\alpha_1\rangle\,|\alpha_2\rangle = \mathrm{e}^{-(|\alpha_1|^2+|\alpha_2|^2/2)} \sum_{A=0}^{\infty}\sum_{k=0}^{A} \frac{\alpha_1^k {\alpha_2}^{A-k}}{\sqrt{k!\,(A-k)!}}\,|k\rangle\,|A-k\rangle \tag{3.51}$$

and introduced the parameterization

$$\alpha_1 = \sqrt{\overline{n}}\cos\frac{\vartheta}{2}\mathrm{e}^{-\mathrm{i}\varphi/2}, \quad \alpha_2 = \sqrt{\overline{n}}\sin\frac{\vartheta}{2}\mathrm{e}^{\mathrm{i}\varphi/2}\mathrm{e}^{-\mathrm{i}\psi} \tag{3.52}$$

we obtain

$$|\alpha_1\rangle\,|\alpha_1\rangle = \sum_{A=0}^{\infty} p_A|\vartheta,\varphi;A\rangle \tag{3.53}$$

where $|\vartheta,\varphi;A\rangle$ is the coherent state (1.17) for A two-level atoms and

$$p_A = \mathrm{e}^{-\overline{n}/2}\frac{\left(\sqrt{\overline{n}}\mathrm{e}^{-\mathrm{i}\psi}\right)^A}{\sqrt{A!}} \tag{3.54}$$

so that $|p_A|^2 = P_A$ is a Poisson distribution. It is clear, that for large value of $\overline{n}$ the expression (3.53) has a sharp maximum at $A = \overline{n}$, so that $|\alpha_1\rangle\,|\alpha_1\rangle \approx \mathrm{e}^{-\mathrm{i}\psi\overline{n}}|\vartheta,\varphi;\overline{n}\rangle$.

3.4
Displacement Operator

The operator

$$D(\alpha) = \exp\left[\alpha a^\dagger - \alpha^* a\right], \quad D^{-1}(\alpha) = D^\dagger(\alpha) = D(-\alpha) \tag{3.55}$$

where α is a complex number, is called the *displacement operator*. The operators $a, a^\dagger, I$ are generators of the Heisenberg–Weyl algebra (see Appendix 11.1) and obey the following commutation relations:

$$\left[a, a^\dagger\right] = I, \quad [a, I] = 0, \quad \left[a^\dagger, I\right] = 0 \tag{3.56}$$

We can express $D(\alpha)$ in a disentangled form (see Appendix 11.2), that is,

$$D(\alpha) = c\mathrm{e}^{\beta a^\dagger}\mathrm{e}^{\gamma a} \tag{3.57}$$

To find the coefficients c, β, γ we introduce a dummy parameter t, according to

$$D(\alpha, t) = \exp\left[t(\alpha a^\dagger - \alpha^* a)\right] = c(t)\,\mathrm{e}^{\beta(t)a^\dagger}\mathrm{e}^{\gamma(t)a} \tag{3.58}$$

so that the value of the right hand side of the above equation at $t = 1$ gives us the desired result. Differentiating both parts with respect to the parameter t,

$$\left(\alpha a^\dagger - \alpha^* a\right)D(\alpha, t) = \dot{c}c^{-1}D(\alpha, t) + \dot{\beta}a^\dagger D(\alpha, t) + c\dot{\gamma}\mathrm{e}^{\beta a^\dagger}a\mathrm{e}^{\gamma a} \tag{3.59}$$

and taking into account that

$$e^{\beta a^\dagger} a e^{-\beta a^\dagger} = a + \beta \left[a^\dagger, a \right] = a - \beta \tag{3.60}$$

we obtain

$$\alpha a^\dagger - \alpha^* a = \dot{c} c^{-1} + \dot{\beta} a^\dagger + \dot{\gamma} \left(a - \beta \right) \tag{3.61}$$

The operators a, $a^\dagger$, and I are linearly independent, so we can extract from Equation 3.61 a system of ordinary differential equations:

$$\dot{c} c^{-1} = \dot{\gamma} \beta, \quad \alpha = \dot{\beta}, \quad -\alpha^* = \dot{\gamma} \tag{3.62}$$

$$c\left(0\right) = 1, \quad \beta\left(0\right) = \gamma\left(0\right) = 0 \tag{3.63}$$

with solution

$$\beta = \alpha t, \ \gamma = -\alpha^* t, \ c = \exp\left[-\frac{|\alpha|^2 t^2}{2} \right] \tag{3.64}$$

finally leading to

$$D\left(\alpha, t = 1\right) = D\left(\alpha\right) = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a} \tag{3.65}$$

Using this form, it is easy to apply the operator $D(\alpha)$ to the vacuum state $|0\rangle$, and since $a|0\rangle = 0$, we have

$$D(\alpha)|0\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} a^{\dagger n} |0\rangle \tag{3.66}$$

Considering that $a^{\dagger n}|0\rangle = \sqrt{n!}|n\rangle$, we immediately obtain

$$D(\alpha)|0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = |\alpha\rangle \tag{3.67}$$

i.e. the application of the displacement operator to the vacuum state generates a coherent state.

Using the disentangled form of the displacement operator and the commutation relation,

$$e^{\beta^* a} e^{\alpha a^\dagger} = e^{\alpha \beta^*} e^{\alpha a^\dagger} e^{\beta a} \tag{3.68}$$

we obtain the composition rule:

$$D(\alpha) D(\beta) = \exp\left[\mathrm{i}\,\mathrm{Im}(\alpha \beta^*) \right] D(\alpha + \beta) \tag{3.69}$$

It is easy to find the following transformation relations:

$$D^{-1}(\alpha) a D(\alpha) = a + \alpha, \quad D^{-1}(\alpha) a^\dagger D(\alpha) = a^\dagger + \alpha^* \tag{3.70}$$

and the matrix element $\langle\beta|D(\gamma)|\alpha\rangle$,

$$\begin{aligned}\langle\beta|D(\gamma)|\alpha\rangle &= \mathrm{e}^{-|\gamma|^2/2}\mathrm{e}^{\gamma\beta^*-\gamma*\alpha}\langle\beta|\alpha\rangle\\ &= \exp\left[-\frac{|\alpha|^2+|\gamma|^2+|\beta|^2}{2}-\gamma^*\alpha+\gamma\beta^*+\alpha\beta^*\right]\end{aligned} \tag{3.71}$$

Using the above formula, it is easy to calculate the trace of the operator $D(\gamma)$. Using the representation

$$\frac{1}{\pi^2}\int \mathrm{d}^2\alpha\exp\left[\gamma\alpha^*-\gamma^*\alpha\right]=\delta^{(2)}(\gamma) \tag{3.72}$$

of the $\delta^{(2)}$ function, we obtain

$$\mathrm{Tr}\,D(\gamma)=\frac{1}{\pi}\int \mathrm{d}^2\alpha\,\langle\alpha|D(\gamma)|\alpha\rangle=\pi\delta^{(2)}(\gamma)=\pi\delta(\mathrm{Re}\,\gamma)\delta(\mathrm{Im}\,\gamma) \tag{3.73}$$

Likewise, from Equation 3.69, we obtain

$$\mathrm{Tr}\left[D(\gamma)D^{-1}(\beta)\right]=\pi\delta^{(2)}(\gamma-\beta) \tag{3.74}$$

Equation 3.73 allows to expand an arbitrary operator $\hat{f}$ over the operational basis of $D(\gamma)$:

$$\hat{f}=\int \mathrm{d}^2\gamma D(-\gamma)\mathrm{Tr}\left(D(\gamma)\hat{f}\right) \tag{3.75}$$

The displacement operator can be also represented in the form of an expansion over the coherent state basis as follows:

$$D(\gamma)=\mathrm{e}^{|\gamma|^2/2}\frac{1}{\pi}\int \mathrm{d}^2\alpha\mathrm{e}^{\gamma\alpha^*-\gamma^*\alpha}|\alpha\rangle\langle\alpha| \tag{3.76}$$

and inversely,

$$|\alpha\rangle\langle\alpha|=\frac{1}{\pi}\int \mathrm{d}^2\gamma\mathrm{e}^{\gamma^*\alpha-\gamma\alpha^*-|\gamma|^2/2}D(\gamma) \tag{3.77}$$

Using the property (3.26) and the generating function for the Laguerre polynomials,

$$\begin{aligned}\exp\left[ax_1+bx_2-ab\right] &= \sum_{n=0}^{\infty}\frac{(ax_1)^n}{n!}\left(1-\frac{b}{x_1}\right)^n\exp\left[\frac{b}{x_1}x_1x_2\right]\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m\frac{a^nb^m}{n!}x_1^{n-m}L_m^{n-m}(x_1x_2)\end{aligned} \tag{3.78}$$

we can obtain from Equation 3.71 the matrix elements of the displacement operator in the number state basis:

$$D_{mn}(\gamma)=\langle m|D(\gamma)|n\rangle=\begin{cases}\sqrt{\dfrac{n!}{m!}}\gamma^{m-n}\mathrm{e}^{-|\gamma|^2/2}L_n^{m-n}\left(|\gamma|^2\right) & m\geq n\\ \sqrt{\dfrac{m!}{n!}}\left(-\gamma^*\right)^{n-m}\mathrm{e}^{-|\gamma|^2/2}L_m^{n-m}\left(|\gamma|^2\right) & n\geq m\end{cases} \tag{3.79}$$

These matrix elements satisfy the following orthogonality relation:

$$\frac{1}{\pi}\int d^2\gamma D_{mn}(\gamma)D^{\dagger}_{m'n'}(\gamma) = \delta_{mn'}\delta_{nm'} \tag{3.80}$$

3.5
Squeezed States

We have already mentioned that for the coherent states, the Heisenberg uncertainty relation is minimized and, also, that the fluctuations of both the canonical variables p and q are equal, $\sigma_q = \sigma_p = 1/2$. In the literature, these values for σ_p and σ_q are referred to as the *standard quantum limit*. In principle, we can look for the states that also minimize the uncertainty relation, but for which $\sigma_q \neq \sigma_p$. Thus, in these states the fluctuations of one of the canonical variables could be lower than the standard quantum limit at the expense of increased fluctuations of the conjugate variable. To find states with this property, let us perform a transformation of the field creation and annihilation operators

$$b = ua + va^{\dagger}, \quad b^{\dagger} = v^*a + u^*a^{\dagger}, \tag{3.81}$$

where u and v are complex numbers. To preserve the commutation relations

$$\left[b, b^{\dagger}\right] = \left[a, a^{\dagger}\right] = 1 \tag{3.82}$$

the parameters u and v should be restricted by

$$|u|^2 - |v|^2 = 1 \tag{3.83}$$

The transformation (3.81) can be rewritten in the matrix form as

$$\begin{bmatrix} b \\ b^{\dagger} \end{bmatrix} = U\begin{bmatrix} a \\ a^{\dagger} \end{bmatrix}, \quad U = \begin{bmatrix} u & v \\ v^* & u^* \end{bmatrix} \tag{3.84}$$

where the parameters u and v satisfy the condition (3.83). Thus, the matrix U is an element of the $SU(1,1)$ group (see Appendix 11.1).

We can introduce the transformation (3.81) from another point of view. It can easily be seen that the operators

$$K_0 = \frac{aa^{\dagger} + a^{\dagger}a}{4}, \quad K_+ = \frac{1}{2}a^{\dagger 2}, \quad K_- = \frac{1}{2}a^2 \tag{3.85}$$

satisfy the commutation relations (11.12)

$$[K_0, K_{\pm}] = \pm K_{\pm}, \quad [K_-, K_+] = 2K_0 \tag{3.86}$$

and, thus, are generators of the $su(1,1)$ algebra. The set of number states is reducible under the action of operators (3.85). The number states decompose into two invariant subspaces, spanned by even and odd number states, respectively. In terms of harmonic oscillator wave functions, the Hilbert space of harmonic oscillator states reduces into invariant subspaces of even or odd parity. The first invariant space is spanned by even number states $|2n\rangle, n = 0, 1, \ldots$, while the other invariant space is spanned by odd number states $|2n+1\rangle, n = 0, 1, \ldots$.

The operators $K_0, K_\pm$ generate transformations g of the type

$$g\begin{bmatrix} a \\ a^\dagger \end{bmatrix} g^{-1} = \begin{bmatrix} gag^{-1} \\ ga^\dagger g^{-1} \end{bmatrix} = \begin{bmatrix} b \\ b^\dagger \end{bmatrix} \tag{3.87}$$

where

$$g = \exp\left[\mathrm{i}\phi_0 K_0 - \eta K_+ + \eta^* K_-\right] \tag{3.88}$$

automatically preserving the property (3.83). It is obvious that the transformation generated by the operator K_0 is reduced to multiplication by a phase

$$\mathrm{e}^{\mathrm{i}\phi K_0}\begin{bmatrix} a \\ a^\dagger \end{bmatrix}\mathrm{e}^{-\mathrm{i}\phi K_0} = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\phi/2}a \\ \mathrm{e}^{\mathrm{i}\phi/2}a^\dagger \end{bmatrix} \tag{3.89}$$

while the Hermitian combinations of the operators $K_\pm$ mix together the operators a and $a^\dagger$:

$$S(\eta)\begin{bmatrix} a \\ a^\dagger \end{bmatrix} S^\dagger(\eta) = \begin{bmatrix} ua + va^\dagger \\ v^*a + u^*a^\dagger \end{bmatrix} = \begin{bmatrix} b \\ b^\dagger \end{bmatrix} \tag{3.90}$$

where

$$S(\eta) = \exp\left[-\eta K_+ + \eta^* K_-\right], \quad \eta = r\mathrm{e}^{\mathrm{i}\theta}, \quad S^\dagger = S^{-1} = S(-\eta) \tag{3.91}$$

The complex numbers u and v are determined using the decomposition formulae (11.29). This yields a parameterization in terms of hyperbolic sine and cosine:

$$u = \cosh r, \ v = \mathrm{e}^{\mathrm{i}\theta}\sinh r \tag{3.92}$$

Now, as we did for the coherent states, we determine the eigenstates of the new annihilation operator b

$$b|\widetilde{\beta}\rangle = \beta|\widetilde{\beta}\rangle \tag{3.93}$$

It can easily be seen that this state has the form

$$|\widetilde{\beta}\rangle = S(\eta)|\beta\rangle = S(\eta)D(\beta)|0\rangle \equiv |\eta, \beta\rangle \tag{3.94}$$

where $|\beta\rangle$ is a coherent state and the state $|\widetilde{\beta}\rangle$ depends on two complex parameters β and η. Using Equation 3.90 and the properties of the displacement operator $D(\alpha) = \exp\left[\alpha a^\dagger - \alpha^* a\right]$, we obtain

$$b|\eta, \beta\rangle = bS(\eta)|\beta\rangle = S(\eta)aD(\beta)|0\rangle = \beta|\eta, \beta\rangle \tag{3.95}$$

Another equivalent representation for the state $|\widetilde{\beta}\rangle$ is

$$|\widetilde{\beta}\rangle = D(\alpha)S(\eta)|0\rangle \equiv |\alpha, \eta\rangle \tag{3.96}$$

where the relation between the parameters β and α is obtained from the following calculations:

$$\begin{aligned} b|\alpha, \eta\rangle &= bD(\alpha)S(\eta)|0\rangle = D(\alpha)\left[u(a+\alpha) + v(a^\dagger + \alpha^*)\right]S(\eta)|0\rangle \\ &= D(\alpha)S(\eta)\left[u\alpha + v\alpha^* + a\right]|0\rangle = \beta|\alpha, \eta\rangle \end{aligned} \tag{3.97}$$

so that

$$\beta = u\alpha + v\alpha^* = \alpha \cosh r + \alpha^* e^{i\theta} \sinh r \tag{3.98}$$

The average values and fluctuations of the canonical variables p and q in the state $|\widetilde{\beta}\rangle = |\alpha, \eta\rangle$ can be easily found by making use Equation 3.90:

$$\langle q \rangle = \frac{1}{\sqrt{2}} \langle \alpha, \eta | a + a^\dagger | \alpha, \eta \rangle = \sqrt{2}\, \mathrm{Re}\, \alpha \tag{3.99}$$

$$\langle p \rangle = \frac{1}{i\sqrt{2}} \langle \alpha, \eta | a - a^\dagger | \alpha, \eta \rangle = \sqrt{2}\, \mathrm{Im}\, \alpha \tag{3.100}$$

$$\sigma_q = \frac{1}{2} |u - v|^2, \quad \sigma_p = \frac{1}{2} |u + v|^2 \tag{3.101}$$

For these states, the covariance σ_{pq} in the states $|\widetilde{\beta}\rangle$ is different from zero,

$$\sigma_{pq} = \mathrm{Im}\left(uv^*\right) \tag{3.102}$$

Thus, the states $|\widetilde{\beta}\rangle$ minimize the Schrödinger–Robertson uncertainty relation (3.31). It is convenient to express the above formulae in terms of real parameters r and θ given by the parameterization (3.92),

$$\sigma_q = \frac{1}{2} \left(\cosh 2r - \cos\theta \sinh 2r\right) \tag{3.103}$$

$$\sigma_p = \frac{1}{2} \left(\cosh 2r + \cos\theta \sinh 2r\right) \tag{3.104}$$

$$\sigma_{pq} = -\frac{1}{2} \sin\theta \sinh 2r \tag{3.105}$$

This implies that the fluctuations of some of the canonical variables may be lower than the standard quantum limit 1/2. The states $|\widetilde{\beta}\rangle$ are then called *squeezed states* and the operator

$$S(\eta) = \exp\left[-\frac{\eta}{2} a^{\dagger 2} + \frac{\eta^*}{2} a^2\right] \tag{3.106}$$

is called the *squeezing operator*.

It can be seen from Equations 3.103 to 3.105, that for $r > 0$, the fluctuations of the variable q are minimized when $\theta = 2\pi n$, which yields

$$\sigma_q = \frac{1}{2} e^{-2r}, \quad \sigma_p = \frac{1}{2} e^{2r}, \quad \sigma_{pq} = 0 \tag{3.107}$$

The states are then squeezed in the q direction.

In the case where $\theta = \pi(2n + 1)$, we have squeezing in the p direction, with

$$\sigma_q = \frac{1}{2} e^{2r}, \quad \sigma_p = \frac{1}{2} e^{-2r}, \quad \sigma_{pq} = 0 \tag{3.108}$$

Finally, if $\theta = \pi(2n + 1/2)$, the covariance σ_{pq} reaches it maximum value (for fixed r),

$$\sigma_q = \sigma_p = \frac{1}{2}\cosh 2r, \quad \sigma_{pq} = \frac{(-1)^{n+1}}{2}\sinh 2r \tag{3.109}$$

and squeezing is observed along the line $q' = p - q$.

Squeezed states have a number of interesting and uncommon properties. We discuss here in some details the so-called squeezed vacuum (see also Problem 4), generated from the normal vacuum by applying the squeezing operator $S(\eta)$:

$$|0, \eta\rangle = S(\eta)|0\rangle \tag{3.110}$$

so that

$$b|0, \eta\rangle = 0 \tag{3.111}$$

Using the Gauss decomposition formulae for the $SU(1, 1)$ group (11.29),

$$S(\eta) = \exp[-\xi K_+]\exp[\xi_0 K_0]\exp[\xi^* K_-] \tag{3.112}$$

where

$$\xi = \mathrm{e}^{\mathrm{i}\theta}\tanh r, \quad \xi_0 = -2\ln(\cosh r) \tag{3.113}$$

and the definition (3.85) of the generators of the $SU(1, 1)$ group, we obtain

$$|0, \eta\rangle = (\cosh r)^{-1/2}\exp\left[-\xi a^{\dagger 2}/2\right]|0\rangle \tag{3.114}$$

which yields the expansion for the squeezed vacuum over the number state basis

$$|0, \eta\rangle = \left(\cosh r\right)^{-1/2}\sum_{n=0}^{\infty}\left[-\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}\tanh r\right]^n \frac{\sqrt{(2n)!}}{n!}|2n\rangle \tag{3.115}$$

It is easy to observe that in this expansion only even number states appear, which means that in the squeezed vacuum the photons would be registered in pairs and the photon distribution has an oscillatory behavior:

$$W_{2n+1} = 0, \quad W_{2n} = |\langle n|0, \eta\rangle|^2 = (\cosh r)^{-1}\left[\frac{\tanh r}{2}\right]^{2n}\frac{(2n)!}{(n!)^2} \tag{3.116}$$

The average number of photons in the squeezed vacuum is

$$\langle n\rangle = \sum_{n=0}^{\infty} W_{2n} n = \sinh^2 r \tag{3.117}$$

which can be large for the values $r \gg 1$.

3.6
Thermal States

The thermal states are defined as those states that maximize the entropy of the field under the condition of constant average energy. The field density matrix, which describes the state of a quantum field in thermal equilibrium with a cavity at the temperature T, has the form

$$\rho = \frac{1}{Z} \exp\left(-\beta a^\dagger a\right) \tag{3.118}$$

where $\beta = 1/kT$ is the inverse temperature (here k is the Boltzmann constant), $\hat{n} = a^\dagger a$ and Z is the partition function, which plays the role of a normalization factor here:

$$Z = Tr\left[\mathrm{e}^{-\beta a^\dagger a}\right] = \sum_n \langle n|\mathrm{e}^{-\beta a^\dagger a}|n\rangle = \sum_n \mathrm{e}^{-\beta n} = \left[1 - \mathrm{e}^{-\beta}\right]^{-1} \tag{3.119}$$

Considering that the average number of photons in the field is

$$\overline{n} = Tr\left(\hat{n}\rho\right) = \frac{1}{Z}\sum_n n\mathrm{e}^{-\beta n} = -\frac{1}{Z}\partial_\beta \sum_n \mathrm{e}^{-\beta n} = \left[\mathrm{e}^{\beta} - 1\right]^{-1} \tag{3.120}$$

we rewrite the density matrix (3.118) as

$$\rho = \frac{1}{1+\overline{n}} \sum_n q^n |n\rangle\langle n| \tag{3.121}$$

where

$$q = \mathrm{e}^{-\beta} = \frac{\overline{n}}{1+\overline{n}} \tag{3.122}$$

Note that the density matrix for the thermal field is diagonal (in the basis of the number states) and, thus, describes a completely mixed state of the field.

It is easy to find the fluctuations of the number of photons in the thermal state; using the obvious relation

$$\overline{n^2} = \frac{1}{Z}\partial_\beta^2 \sum_n \mathrm{e}^{-\beta n} = \overline{n}\left(2\overline{n}+1\right) \tag{3.123}$$

we obtain

$$\sigma_n^2 = \overline{n}^2 + \overline{n} \tag{3.124}$$

3.7
Phase Operator

Several different approaches have been considered to investigate the correct construction and interpretation of a quantum phase operator. The simplest but ultimately incomplete method was proposed by Dirac. The method attempts to decompose the field creation and annihilation operators in polar form, i.e. to write these operators as a product of a Hermitian, a positive definite operator, and a unitary operator: $\hat{f} = \hat{h}\hat{U}$, where $\hat{h} = \sqrt{\hat{f}\hat{f}^\dagger}$ and $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = I$. It turns out that this approach faces serious practical difficulties. To illustrate this last point, let us represent the annihilation and creation operators $a, a^\dagger$ in the following polar form:

$$a = \sqrt{aa^\dagger}\mathrm{e}^{\mathrm{i}\hat{\phi}} \quad a^\dagger = \sqrt{a^\dagger a}\mathrm{e}^{-\mathrm{i}\hat{\phi}} \tag{3.125}$$

Then it is easy to verify that in the basis of number states, the operators $\mathrm{e}^{\pm\mathrm{i}\hat{\phi}}$ act as follows

$$\mathrm{e}^{\mathrm{i}\hat{\phi}}|n\rangle = \frac{1}{\sqrt{aa^\dagger}}a|n\rangle = \begin{cases} 0, & n = 0 \\ |n-1\rangle, & n \neq 0 \end{cases} \tag{3.126}$$

and

$$\mathrm{e}^{-\mathrm{i}\hat{\phi}}|n\rangle = a^\dagger\frac{1}{\sqrt{aa^\dagger}}|n\rangle = |n+1\rangle \tag{3.127}$$

Thus, the operators $\mathrm{e}^{\mathrm{i}\hat{\phi}}$ and $\mathrm{e}^{-\mathrm{i}\hat{\phi}}$ are not unitary, since

$$\mathrm{e}^{\mathrm{i}\hat{\phi}}\mathrm{e}^{-\mathrm{i}\hat{\phi}} = I, \quad \mathrm{e}^{-\mathrm{i}\hat{\phi}}\mathrm{e}^{\mathrm{i}\hat{\phi}} = I - |0\rangle\langle 0| \tag{3.128}$$

i.e.

$$\mathrm{e}^{-\mathrm{i}\hat{\phi}}\mathrm{e}^{\mathrm{i}\hat{\phi}}|n\rangle = \begin{cases} 0, & n = 0 \\ |n\rangle, & n \neq 0 \end{cases}$$

so that

$$\left[\mathrm{e}^{\mathrm{i}\hat{\phi}}, \mathrm{e}^{-\mathrm{i}\hat{\phi}}\right] = |0\rangle\langle 0| \tag{3.129}$$

The problem with the polar decomposition arises from the fact that neither a nor $a^\dagger$ can be inverted and are, in this sense, degenerated. From the physical point of view, this is related with the semiboundedness of the Hilbert space.

In spite of this difficulty, the decomposition is useful in many situations in which the amplitude of the vacuum is small. In such situations, the action of the phase operators $\mathrm{e}^{\pm\mathrm{i}\hat{\phi}}$ is "almost" unitary.

From the formal definition of the phase operators [27],

$$\mathrm{e}^{-\mathrm{i}\hat{\phi}} = a^\dagger\frac{1}{\sqrt{aa^\dagger}} = \sum_{n=0}^{\infty}|n+1\rangle\langle n|, \quad \mathrm{e}^{\mathrm{i}\hat{\phi}} = \frac{1}{\sqrt{aa^\dagger}}a = \sum_{n=0}^{\infty}|n\rangle\langle n+1| \tag{3.130}$$

we infer the following commutators:

$$\left[a^\dagger a, \mathrm{e}^{\mathrm{i}\hat{\phi}}\right] = -\mathrm{e}^{\mathrm{i}\hat{\phi}}, \quad \left[a^\dagger a, \mathrm{e}^{-\mathrm{i}\hat{\phi}}\right] = \mathrm{e}^{-\mathrm{i}\hat{\phi}} \tag{3.131}$$

and, in general, for any function of the operator $\hat{n} = a^\dagger a$, we have

$$f(\hat{n})\mathrm{e}^{\mathrm{i}\hat{\phi}} = \mathrm{e}^{\mathrm{i}\hat{\phi}}f(\hat{n}-1), \quad f(\hat{n})\mathrm{e}^{-\mathrm{i}\hat{\phi}} = \mathrm{e}^{-\mathrm{i}\hat{\phi}}f(\hat{n}+1) \tag{3.132}$$

It is easy to prove these relations by using the Taylor series expansion of the function $f(\hat{n})$.

It is useful to introduce Hermitian combinations of the phase operators [28],

$$\cos\hat{\phi} = \frac{e^{i\hat{\phi}} + e^{-i\hat{\phi}}}{2}, \quad \sin\hat{\phi} = \frac{e^{i\hat{\phi}} - e^{-i\hat{\phi}}}{2i} \tag{3.133}$$

Since $\exp(i\hat{\phi})^\dagger = \exp(-i\hat{\phi})$, it follows that $\left(\cos\hat{\phi}\right)^\dagger = \cos\hat{\phi}$, $\left(\sin\hat{\phi}\right)^\dagger = \sin\hat{\phi}$, and that

$$\left[\cos\hat{\phi}, \sin\hat{\phi}\right] = \frac{i}{2}|0\rangle\langle 0|$$

It is worth noting that the operators $\cos\hat{\phi}$ and $\sin\hat{\phi}$ are connected through the following relation:

$$\cos^2\hat{\phi} + \sin^2\hat{\phi} = I - \frac{1}{2}|0\rangle\langle 0| \tag{3.134}$$

Let us calculate the average values and fluctuations of the operators $\cos\hat{\phi}$ and $\sin\hat{\phi}$ in the number states,

$$\langle n|\cos\hat{\phi}|n\rangle = \langle n|\sin\hat{\phi}|n\rangle = 0 \tag{3.135}$$

$$\left\langle n|\cos^2\hat{\phi}|n\right\rangle = \langle n|\sin^2\hat{\phi}|n\rangle = \begin{cases} 1/2, & n \neq 0 \\ 1/4, & n = 0 \end{cases} \tag{3.136}$$

From this, we readily obtain the fluctuations

$$\sigma_{\cos\hat{\phi}} = \langle n|\cos^2\hat{\phi}|n\rangle - \langle n|\cos\hat{\phi}|n\rangle^2 = \frac{1}{2}, \quad n \neq 0 \tag{3.137}$$

Another interesting quantity is the phase fluctuation in the coherent states [28],

$$\langle\alpha|\cos\hat{\phi}|\alpha\rangle = \frac{1}{2}e^{-|\alpha|^2}\sum_{m,n}\frac{\alpha^n\alpha^{*m}}{\sqrt{n!m!}}\langle m|e^{i\hat{\phi}} + e^{-i\hat{\phi}}|n\rangle \tag{3.138}$$

Since

$$\langle m|e^{i\hat{\phi}}|n\rangle = \delta_{m,n+1}, \quad \langle m|e^{-i\hat{\phi}}|n\rangle = \delta_{m,n-1} \tag{3.139}$$

we obtain

$$\langle\alpha|\cos\hat{\phi}|\alpha\rangle = e^{-|\alpha|^2}\,\mathrm{Re}\,\alpha\sum_{n=0}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}} \tag{3.140}$$

or, writing $\alpha = \sqrt{\overline{n}}e^{i\varphi}$, we have

$$\langle\alpha|\cos\hat{\phi}|\alpha\rangle = e^{-\overline{n}}\sqrt{\overline{n}}\cos\varphi\sum_{n}\frac{\overline{n}^n}{n!\sqrt{n+1}} \tag{3.141}$$

This sum cannot be further simplified. However, in the limit of intense fields, where $\overline{n} \gg 1$, the sum can be evaluated approximately. Using the following integral representation

$$\frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{d}z \, \frac{\mathrm{e}^{-z(n+1)}}{\sqrt{z}} \tag{3.142}$$

we obtain

$$\sum_n \frac{\overline{n}^n}{n!\sqrt{n+1}} = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{d}z \frac{\mathrm{e}^{-z}}{\sqrt{z}} \sum_n \frac{(\overline{n}\mathrm{e}^{-z})^n}{n!} = \frac{1}{\sqrt{\pi}} \int_0^1 \mathrm{d}x \frac{\mathrm{e}^{\overline{n}(1-x)}}{\sqrt{-\ln\,(1-x)}} \tag{3.143}$$

where in the last integral we changed the variable $\mathrm{e}^{-z} = 1 - x$. Note that in the case of $\overline{n} \gg 1$ the characteristic width of the function being integrated $x \sim \overline{n}^{-1} \ll 1$ and, thus, we can expand $\ln(1-x)$ as a power series

$$\sqrt{-\ln\,(1-x)} = \sqrt{x + x^2/2 + \cdots} = \sqrt{x}\left(1 + \frac{x}{4}\right) + \cdots \tag{3.144}$$

By substituting this in the integral (3.143), we obtain

$$\begin{aligned}\frac{1}{\sqrt{\pi}} \int_0^1 \mathrm{d}x \frac{\mathrm{e}^{-\overline{n}x}}{\sqrt{-\ln\,(1-x)}} &= \frac{1}{\sqrt{\pi}} \int_0^1 \mathrm{d}x \frac{\mathrm{e}^{-\overline{n}x}}{\sqrt{x}} \left(1 - \frac{x}{4} + \cdots\right) \\ &= \frac{1}{\sqrt{\overline{n}}} - \frac{1}{8\overline{n}^{3/2}} + \cdots \end{aligned} \tag{3.145}$$

and, finally,

$$\langle\alpha|\cos\hat{\phi}|\alpha\rangle = \cos\varphi\left(1 - \frac{1}{8\overline{n}} + \cdots\right) \tag{3.146}$$

To find the fluctuation of the phase in the coherent state, we need to calculate $\langle\alpha|\cos^2\hat{\phi}|\alpha\rangle$. Likewise, expanding the phase fluctuation (3.138) we obtain

$$\langle\alpha|\cos^2\hat{\phi}|\alpha\rangle = \frac{1}{2} - \frac{1}{4}\mathrm{e}^{-\overline{n}} + \overline{n}\left(\cos^2\varphi - \frac{1}{2}\right)\mathrm{e}^{-\overline{n}} \sum_n \frac{\overline{n}^n}{n!\sqrt{(n+1)\,(n+2)}} \tag{3.147}$$

The function in the last sum is maximum for $n \sim \overline{n} \gg 1$, thus we can represent the root as

$$\frac{1}{\sqrt{(n+1)\,(n+2)}} = \frac{1}{\sqrt{(n+1)^2 + (n+1)}} \simeq \frac{1}{n+1} - \frac{1}{2\,(n+1)^2} + \cdots \tag{3.148}$$

A simple calculation then yields

$$\langle\alpha|\cos^2\hat{\phi}|\alpha\rangle = \cos^2\varphi - \frac{\cos^2\varphi - 1/2}{2\overline{n}} + \cdots \tag{3.149}$$

It is easy to find similar expressions for $\langle\alpha|\sin^2\hat{\phi}|\alpha\rangle$. Thus, the fluctuations of the phase in the coherent state for $\overline{n} \gg 1$ have the form

$$\sigma_{\cos\hat{\phi}} \simeq \frac{\sin^2\varphi}{4\overline{n}} \tag{3.150}$$

which tends to zero when $\overline{n} \to \infty$ (observe that $\sigma^2_{\sin\hat{\phi}} \simeq \cos^2\varphi/4\overline{n}$ due to Equation 3.134). This means that the phase in highly excited coherent states is quite well defined:

$$e^{\pm i\hat{\phi}}|\alpha\rangle = e^{\pm i\varphi}|\alpha\rangle + O\left(\frac{1}{\sqrt{\overline{n}}}\right) \tag{3.151}$$

Eigenstates for the phase operator are introduced as

$$|\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{in\theta}|n\rangle \tag{3.152}$$

and $a^\dagger a|\theta\rangle = \sum_n e^{in\theta} n|n\rangle = -i\partial_\theta|\theta\rangle$.

One can easily verify the resolution of the identity

$$\int_{\theta_0}^{\theta_0+2\pi} d\theta|\theta\rangle\langle\theta| = I \tag{3.153}$$

so that

$$e^{\pm i\hat{\phi}} = \int_{\theta_0}^{\theta_0+2\pi} d\theta|\theta\rangle\langle\theta|e^{\pm i\theta}$$

where θ_0 defines the integration window.

The states (3.152) are normalized according to [29]

$$\langle\theta|\theta'\rangle = \frac{1}{2\pi}\sum_{n=0}^{\infty} e^{in(\theta'-\theta)} = \frac{1}{2}\delta(\theta-\theta') + \frac{1}{4\pi}\left[1 + i\cot\frac{(\theta'-\theta)}{2}\right] \tag{3.154}$$

Note that in the infinite-dimensional space the basis elements (number states) are represented as vectors

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \ldots \tag{3.155}$$

and the phase operators defined in by Equations 3.126 and 3.127 are matrices

$$e^{-i\hat{\phi}} = \begin{bmatrix} 0 & * & * & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \quad e^{i\hat{\phi}} = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ * & 0 & 1 & \cdots \\ * & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{3.156}$$

where the elements marked as $(*)$ of the first column or the first row ($(e^{i\hat{\phi}})^\dagger = e^{-i\hat{\phi}}$) are not determined. These matrix elements are often set to 0.

The matrices $e^{i\hat{\phi}}$ and $e^{-i\hat{\phi}}$ do not commute: $\left[e^{i\hat{\phi}}, e^{-i\hat{\phi}}\right] = |0\rangle\langle 0|$, according to Equation 3.129. Another definition of states $|\theta\rangle$ is discussed in the following section using regularization in finite-dimensional space.

As a final remark, we believe that the most natural way of dealing with quantum phases would be to use positive operator-valued measures (POVM), $\hat{\Delta}(\varphi)$. They are defined to produce measurable probabilities $p(\varphi)$ for a given field state with the density matrix ρ: $p(\varphi) = Tr\left(\hat{\Delta}(\varphi)\rho\right)$. Such POVM must also satisfy the following relations:

$$\hat{\Delta}(\varphi) = \hat{\Delta}^\dagger(\varphi), \qquad \int_{\varphi_0}^{\varphi_0+2\pi} d\varphi \hat{\Delta}(\varphi) = I$$

The physical condition of complementarity between the phase and the photon number $\hat{n}$ impose that the argument of $\hat{\Delta}(\varphi)$ should be displaced under the action of an unitary operator generated by $\hat{n}$:

$$e^{i\varphi'\hat{n}}\hat{\Delta}(\varphi)e^{-i\varphi'\hat{n}} = \hat{\Delta}(\varphi + \varphi')$$

Nevertheless, in this book we do not use the POVM techniques and we refer the reader to specialized articles devoted to this interesting question [30].

3.8
Regularized Phase Operator

The Hilbert space of a single field mode is isomorphic to that of the harmonic oscillator and thus is infinite dimensional. The definition of phase operators is plagued by problems associated with the action of phase operators on vacuum. To bypass these, it was proposed [31] to first define phase operators in a finite-dimensional space of dimension $s+1$, keeping s as a parameter and expressing all calculations in terms of s, and then taking the limit $s \to \infty$. Following this procedure, we start with a reference state $|\theta_0\rangle$

$$|\theta_0\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{in\theta_0}|n\rangle \tag{3.157}$$

and construct a basis in the s-dimensional space according to

$$|\theta_m\rangle = \exp\left[\frac{2\pi i m}{s+1}\hat{n}\right]|\theta_0\rangle, \quad m = 0, 1, 2, \ldots, s \tag{3.158}$$

Thus, in the number state basis (in the finite-dimensional space) the states $|\theta_m\rangle$ are represented as

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp\left[in\left(\theta_0 + \frac{2m\pi}{s+1}\right)\right]|n\rangle \tag{3.159}$$

or

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp(\mathrm{i}n\theta_m)|n\rangle \tag{3.160}$$

where the parameter θ_m is

$$\theta_m = \theta_0 + \frac{2m\pi}{s+1}, \quad \theta_0 \le \theta_m \le \theta_0 + \frac{2s\pi}{s+1} \tag{3.161}$$

The regularization method is based on the following result, well known in the theory of the finite Fourier transform:

$$\frac{1}{s+1} \sum_{m=0}^{s} \exp\left[\frac{2\pi\mathrm{i}(n-k)m}{(s+1)}\right] = \delta_{n,k} \tag{3.162}$$

Using this relation, one obtains

$$\langle\theta_k|\theta_n\rangle = \delta_{k,n}, \quad \sum_{m=0}^{s} |\theta_m\rangle\langle\theta_m| = 1 \tag{3.163}$$

which means that the states $|\theta_m\rangle$ form a complete orthogonal basis in finite-dimensional space. Note that the projector $|\theta_m\rangle\langle\theta_m|$, expressed in terms of the states $|n\rangle$ has the form

$$|\theta_m\rangle\langle\theta_m| = \frac{1}{s+1} \sum_{n',n}^{s} \mathrm{e}^{\mathrm{i}\theta_m(n'-n)}|n'\rangle\langle n| \tag{3.164}$$

For example, a number state can be represented as

$$|n\rangle = \frac{1}{\sqrt{s+1}} \sum_{k=0}^{s} \mathrm{e}^{-\mathrm{i}n\theta_k}|\theta_k\rangle \tag{3.165}$$

Now, the *Hermitian* phase operator $\hat{\phi}$ is defined as

$$\hat{\phi}_\theta = \sum_{m=0}^{s} \theta_m|\theta_m\rangle\langle\theta_m| = \theta_0 I + \frac{2\pi}{s+1} \sum_{m=0}^{s} m|\theta_m\rangle\langle\theta_m| \tag{3.166}$$

and depends on the reference phase θ_0. Using Equation 3.163 we immediately obtain

$$\mathrm{e}^{\mathrm{i}\hat{\phi}_\theta} = \exp\left[\mathrm{i}\sum_m \theta_m|\theta_m\rangle\langle\theta_m|\right] = \sum_m \mathrm{e}^{\mathrm{i}\theta_m}|\theta_m\rangle\langle\theta_m| \tag{3.167}$$

so that

$$\mathrm{e}^{\pm\mathrm{i}\hat{\phi}_\theta}|\theta_m\rangle = \mathrm{e}^{\pm\mathrm{i}\theta_m}|\theta_m\rangle \tag{3.168}$$

Thus, in view of Equation 3.165 we find

$$\mathrm{e}^{\mathrm{i}\hat{\phi}_\theta}|n\rangle = \frac{1}{\sqrt{s+1}} \sum_{m=0}^{s} \mathrm{e}^{-\mathrm{i}\theta_m(n-1)}|\theta_m\rangle \tag{3.169}$$

or

$$e^{i\hat{\phi}_\theta}|n\rangle = \begin{cases} |n-1\rangle, & n > 0, \\ \dfrac{1}{\sqrt{s+1}} \displaystyle\sum_{m=0}^{s} e^{i\theta_m}|\theta_m\rangle, & n = 0 \end{cases} \tag{3.170}$$

However, one can rewrite $e^{i\theta_m}$ in the following form:

$$e^{i\theta_m} = e^{i(s+1)\theta_0} \exp\left[i\left(2m\pi - s\theta_0 - \frac{2\pi ms}{s+1}\right)\right] = e^{i(s+1)\theta_0} e^{-is\theta_m} \tag{3.171}$$

which implies that

$$\frac{1}{\sqrt{s+1}} \sum_{m=0}^{s} e^{i\theta_m}|\theta_m\rangle = \frac{e^{i(s+1)\theta_0}}{\sqrt{s+1}} \sum_{m} e^{-is\theta_m}|\theta_m\rangle = e^{i(s+1)\theta_0}|s\rangle \tag{3.172}$$

Finally, the action of the phase operator on the vacuum state is given by

$$e^{i\hat{\phi}_\theta}|0\rangle = e^{i(s+1)\theta_0}|s\rangle \tag{3.173}$$

In the same way, it can be shown that

$$e^{-i\hat{\phi}_\theta}|n\rangle = |n+1\rangle, \quad n \le s, \quad e^{-i\hat{\phi}_\theta}|s\rangle = e^{-i(s+1)\theta_0}|0\rangle \tag{3.174}$$

and, thus

$$e^{i\hat{\phi}_\theta} e^{-i\hat{\phi}_\theta} = e^{-i\hat{\phi}_\theta} e^{i\hat{\phi}_\theta} = 1 \tag{3.175}$$

in the entire $(s+1)$-dimensional space. This implies that the operators $\exp(\pm i\hat{\phi}_\theta)$ are *unitary* in the finite-dimensional space.

The matrix form of the regularized phase operators is thus,

$$e^{-i\hat{\phi}_\theta} = \begin{bmatrix} 0 & 0 & \cdots & e^{-i(s+1)\theta_0} \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad e^{i\hat{\phi}_\theta} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & 1 \\ e^{i(s+1)\theta_0} & 0 & 0 & 0 \end{bmatrix} \tag{3.176}$$

(compare with the corresponding formulae for the operators $\exp(\pm i\hat{\phi})$ in the previous section).

3.9
Phase Distribution

We assume that the field is in an arbitrary state $|\psi\rangle$, which we represent as an expansion over the number states:

$$|\psi\rangle = \sum_{n=0}^{s} C_n|n\rangle \tag{3.177}$$

Using Equation 3.165, we can expand this state over the eigenstates of the phase operator $|\theta_m\rangle$:

$$|\psi\rangle = \sum_{m=0}^{s} |\theta_m\rangle\langle\theta_m|\psi\rangle, \quad \langle\theta_m|\psi\rangle = \frac{1}{\sqrt{s+1}}\sum_{n=0}^{s} C_n \mathrm{e}^{-in\theta_m} \tag{3.178}$$

The distribution function is determined as

$$W_{\theta_m} = |\langle\theta_m|\psi\rangle|^2 = \frac{1}{s+1}\left|\sum_{n=0}^{s} C_n \mathrm{e}^{-in\theta_m}\right|^2 \tag{3.179}$$

and is interpreted as the probability of finding the state $|\psi\rangle$ with the well-defined phase θ_m. Note that, in the finite-dimensional space, the possible values of phases form a discrete set.

The distribution function W_{θ_m} allows us to calculate any average, for example,

$$\langle\psi|\hat{\phi}_\theta|\psi\rangle = \sum_m \theta_m W_{\theta_m}, \quad \langle\psi|\cos\hat{\phi}_\theta|\psi\rangle = \sum_m \cos\theta_m W_{\theta_m} \tag{3.180}$$

We represent the expansion coefficients in the polar form

$$C_n = \mathrm{e}^{in\beta}|C_n| \tag{3.181}$$

The reference phase θ_0 is selected to be

$$\theta_0 = -\frac{s\pi}{s+1} \tag{3.182}$$

Introducing a new index $\kappa = m - s/2$, which runs from $-s/2$ to $s/2$, Equation 3.179 is rewritten as

$$W_{\theta_m} = \frac{1}{s+1}\left|\sum_n C_n \mathrm{e}^{-in\kappa(2\pi/s+1)}\right|^2 \tag{3.183}$$

When the space dimension tends to infinity, $s \to \infty$, this sum becomes a continuous function of the angle, so that

$$\theta_m \longrightarrow \theta = \kappa\frac{2\pi}{s+1}, \quad \frac{2\pi}{s+1} = \mathrm{d}\theta \tag{3.184}$$

Thus the distribution function of the discrete angle W_{θ_m} becomes the probability density W_θ so that

$$W_\theta\,\mathrm{d}\theta = \frac{1}{2\pi}\left|\sum_n C_n \mathrm{e}^{-in\theta}\right|^2 \mathrm{d}\theta = |\langle\theta|\psi\rangle|^2\,\mathrm{d}\theta \tag{3.185}$$

This density is normalized as

$$\int_{-\pi}^{\pi} W_\theta \mathrm{d}\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_n C_n \mathrm{e}^{-in\theta}\right|^2 \mathrm{d}\theta = 1 \tag{3.186}$$

Thus any average related to the phase operator is calculated in the following way:

$$\langle\psi|f(\hat{\phi})|\psi\rangle = \int_{-\pi}^{\pi} f(\theta)\, W_\theta\, \mathrm{d}\theta \tag{3.187}$$

Let us find W_θ for the coherent states $|\alpha\rangle$, $\alpha = \sqrt{\overline{n}}\mathrm{e}^{\mathrm{i}\varphi}$, for the case in which $\overline{n} \gg 1$. From Equation 3.185, we have

$$W_\theta = \frac{1}{2\pi}\left|\sum_n \mathrm{e}^{-\overline{n}/2}\frac{\overline{n}^{n/2}}{\sqrt{n!}}\mathrm{e}^{\mathrm{i}n(\varphi-\theta)}\right|^2 \tag{3.188}$$

We use the following formula to evaluate the above sum:

$$\left|\sum a_k\right|^2 = 2\mathrm{Re}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}{}' a_n \alpha_{n+j}^* \tag{3.189}$$

where ($'$) means that the term with $j = 0$ is taken with coefficient 1/2. Thus, Equation 3.188 can be brought to the form

$$W_\theta = \frac{1}{2\pi}2\,\mathrm{Re}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}{}' p_n p_{n+j}\mathrm{e}^{\mathrm{i}j(\theta-\varphi)}, \tag{3.190}$$

where p_n is the square root of the Poisson distribution: $p_n = \mathrm{e}^{-\overline{n}/2}\overline{n}^{n/2}/\sqrt{n!}$. It follows from the properties of the Poisson distribution that the value of j in the term p_{n+j} in the sum (3.190) is of the order of the width of the Poisson distribution: $j \sim \sqrt{\overline{n}}$. For $n \sim \overline{n} \gg 1$, we can approximate the Poisson distribution by the Gaussian distribution:

$$p_n \approx (2\pi\overline{n})^{-(1/4)}\exp\left[-\frac{(n-\overline{n})^2}{4\overline{n}}\right] \tag{3.191}$$

Thus, p_{n+j} can be approximated as follows:

$$p_{n+j} \approx p_n \exp\left[-\frac{j^2}{4\overline{n}} - \frac{j(n-\overline{n})}{2\overline{n}}\right] \tag{3.192}$$

Substituting this approximation in Equation 3.190, we obtain

$$W_\theta = \frac{1}{2\pi}2\,\mathrm{Re}\sum_{j=0}^{\infty}{}'\mathrm{e}^{\mathrm{i}j(\theta-\varphi)}\mathrm{e}^{-j^2/4\overline{n}}\mathrm{e}^{j/2}\sum_{n=0}^{\infty}p_n^2\exp\left[-\frac{jn}{2\overline{n}}\right] \tag{3.193}$$

$$= \frac{1}{\pi}\mathrm{Re}\sum_{j=0}^{\infty}{}'\mathrm{e}^{\mathrm{i}j(\theta-\varphi)}\mathrm{e}^{-j^2/4\overline{n}}\mathrm{e}^{j/2}\exp\left[\overline{n}\left(\mathrm{e}^{-j/2\overline{n}}-1\right)\right] \tag{3.194}$$

Since $j \sim \sqrt{\overline{n}} \ll \overline{n}$, we can approximate

$$\mathrm{e}^{-j/2\overline{n}} - 1 \simeq -\frac{j}{2\overline{n}} + \frac{j^2}{8\overline{n}^2} \tag{3.195}$$

and, it follows that

$$W_\theta \simeq \frac{1}{\pi}\mathrm{Re}\sum_{j=0}^{\infty}{}' \mathrm{e}^{\mathrm{i}j(\theta-\varphi)} \exp\left[-\frac{j^2}{8\overline{n}}\right] = \frac{1}{2\pi}\sum_{j=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}j(\theta-\varphi)} \exp\left[-\frac{j^2}{8\overline{n}}\right] \tag{3.196}$$

It can easily be seen that W_θ is normalized,

$$\int_{-\pi}^{\pi} W_\theta \mathrm{d}\theta = 1 \tag{3.197}$$

The expression (3.196) can be rewritten in terms of the Jacobi theta functions $\Theta_3(z|\mu)$,

$$\Theta_3(z|\mu) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-2\mathrm{i}zn}\mathrm{e}^{-\mu n^2} \tag{3.198}$$

as

$$W_\theta \simeq \frac{1}{2\pi}\Theta_3(z|\mu), \quad z = \frac{\varphi-\theta}{2}, \quad \mu = \frac{1}{8\overline{n}} \tag{3.199}$$

Applying the Poisson summation formula,

$$\sum_{m=-\infty}^{\infty} f_m = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x \mathrm{e}^{2\pi\mathrm{i}kx} f(x) \tag{3.200}$$

we obtain from Equation 3.196 [32]

$$W_\theta = \frac{1}{2\pi}\sum_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x \mathrm{e}^{\mathrm{i}2kx\pi+\mathrm{i}x(\theta-\varphi)-x^2/8\overline{n}} = \frac{1}{2\pi}\sqrt{8\pi\overline{n}}\sum_{-\infty}^{\infty} \mathrm{e}^{-2\overline{n}(\theta-\varphi-2\pi k)^2} \tag{3.201}$$

or, finally,

$$W_\theta = \frac{1}{\sqrt{2\pi}\sigma}\sum_{-\infty}^{\infty}\exp\left[-\frac{(\theta-\varphi-2\pi k)^2}{2\sigma^2}\right] \tag{3.202}$$

where the dispersion is $\sigma^2 = 1/4\overline{n}$. The above sum contains Gaussians displaced by $2\pi k$, which guarantees the periodicity of W_θ in the angle θ.

Let us calculate the average and the operator fluctuation $\cos\hat{\phi}$ using the form (3.196) for the distribution function.

$$\langle\cos\hat{\phi}\rangle = \int_{-\pi}^{\pi} \mathrm{d}\theta \, \cos\theta \, W_\theta = \frac{1}{2\pi}\sum_{-\infty}^{\infty} \mathrm{e}^{-j^2/8\overline{n}}\mathrm{e}^{-\mathrm{i}j\varphi}\int_{-\pi}^{\pi} \mathrm{d}\theta \, \mathrm{e}^{\mathrm{i}j\theta}\cos\theta \tag{3.203}$$

The integral is easily calculated and yields

$$\langle\cos\hat{\phi}\rangle = \frac{1}{2}\sum_{-\infty}^{\infty} \mathrm{e}^{-j^2/8\overline{n}}\mathrm{e}^{-\mathrm{i}j\varphi}\left[\delta_{j,1}+\delta_{j,-1}\right] = \mathrm{e}^{-1/8\overline{n}}\cos\varphi \tag{3.204}$$

Likewise, we can evaluate $\left\langle \cos^2 \hat{\phi} \right\rangle$:

$$\langle \cos^2 \hat{\phi} \rangle = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{-j^2/8\overline{n}} e^{-ij\varphi} \int_{-\pi}^{\pi} d\theta\, e^{ij\theta} \cos^2\theta \tag{3.205}$$

$$= \frac{1}{4} \sum_{-\infty}^{\infty} e^{-j^2/8\overline{n}} e^{-ij\varphi} \left[\delta_{j,2} + \delta_{j,-2} + 2\right] = \frac{1}{2} e^{-1/2\overline{n}} \cos 2\varphi + \frac{1}{2} \tag{3.206}$$

Finally, for the fluctuation $\sigma_{\cos\phi} = \left\langle \cos^2 \hat{\phi} \right\rangle - \left\langle \cos \hat{\phi} \right\rangle^2$ we have

$$\sigma_{\cos\phi} = \frac{1 - e^{-1/4\overline{n}}}{2} + \frac{\cos 2\varphi}{2} e^{-1/4\overline{n}} \left(e^{-1/4\overline{n}} - 1\right) \simeq \frac{1}{8\overline{n}} - \frac{\cos 2\varphi}{8\overline{n}} = \frac{\sin^2\varphi}{4\overline{n}} \tag{3.207}$$

which coincides with the result obtained in Equation 3.150.

On the contrary to the coherent states the phase distribution of a number state $|N\rangle$ is uniform,

$$W_\theta = \frac{1}{2\pi} |\langle \theta | N \rangle|^2 = \frac{1}{2\pi}$$

3.10
Problems

3.1 The density matrix of a quantum system is given by

$$\rho = a^\dagger e^{-\lambda a^\dagger a} a \tag{3.208}$$

Determine what state describes this density matrix in the limits: (a) $\lambda \to 0$, (b) $\lambda \to \infty$.

3.2 Prove that the state

$$|\psi\rangle = N\left(|\alpha + \delta\rangle + |\alpha - \delta\rangle\right) \tag{3.209}$$

where α and β are real, $\delta \ll \alpha$, and N is the normalization constant, such that $\langle \psi | \psi \rangle = 1$, is a squeezed state.

3.3 Find the phase fluctuation for the state

$$|\psi\rangle = N\left(|\alpha + \delta\rangle + |\alpha - \delta\rangle\right) \tag{3.210}$$

where α and β are real, $\delta \ll \alpha$, and N is the normalization constant, such that $\langle\psi|\psi\rangle = 1$.

3.4 Find the expansion of the squeezed state (3.96) in the number state basis

$$|\alpha, \eta\rangle = \sum_n b_n|n\rangle \tag{3.211}$$

Hint. Make use of the relation (3.97) and obtain the following difference equation for $a_n = b_n/\sqrt{n!}$:

$$ua_{n+1} + vna_{n-1} = \beta a_n \tag{3.212}$$

where β is defined in Equation 3.98, which immediately allows us to connect b_n with the Hermite polynomials.

4
Field Dynamics

In this chapter, we discuss, for completeness, the evolution of a quantized field under action of some typical Hamiltonians. A good deal of attention is paid to the algebraic tools used for studying field dynamics.

4.1
Evolution of a Field with Classical Pumping

The Hamiltonian describing the evolution of a single quantized field mode under the influence of an external force (classical pumping) has the form ($\hbar = 1$)

$$H = \omega a^\dagger a + f(t) a + f(t)^* a^\dagger \tag{4.1}$$

where $f(t)$ represents the pumping field. Because the operators $\{a^\dagger a, a, a^\dagger, I\}$ close an oscillator algebra (see Appendix 11.1), the evolution operator can be written in the form

$$U(t) = C(t) \mathrm{e}^{\alpha(t)\, a^\dagger a} \mathrm{e}^{\beta(t) a^\dagger} \mathrm{e}^{\gamma(t) a} \tag{4.2}$$

where the time-dependent parameters are the solution of a system of ordinary differential equations, obtained after substitution of Equation 4.2 into the Schrödinger equation $\mathrm{i}\dot{U} = HU$, in accordance with the general procedure described in Appendix 11.3. Since the operators $\{a^\dagger a, a, a^\dagger, I\}$ are linearly independent, we obtain the coupled equations

$$\dot{C}/C = \dot{\gamma}\beta, \quad \mathrm{i}\dot{\alpha} = \omega, \quad \mathrm{i}\dot{\beta}\mathrm{e}^{\alpha} = f(t)^*, \quad \mathrm{i}\dot{\gamma}\mathrm{e}^{-\alpha} = f(t) \tag{4.3}$$

satisfying the initial conditions

$$C(0) = 1, \quad \alpha(0) = \beta(0) = \gamma(0) = 0 \tag{4.4}$$

These initial conditions guarantee that $U(0) = I$. This system is easily solved with

$$C(t) = \exp\left[-\delta\right], \quad \alpha(t) = -\mathrm{i}\omega t \tag{4.5}$$

$$\beta(t) = -\mathrm{i}\int_0^t \mathrm{d}\tau \mathrm{e}^{\mathrm{i}\omega\tau} f^*(\tau), \quad \gamma(t) = -\mathrm{i}\int_0^t \mathrm{d}\tau \mathrm{e}^{-\mathrm{i}\omega\tau} f(\tau) \tag{4.6}$$

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where

$$\delta(t) = \int_0^t d\tau e^{-i\omega\tau} f(\tau) \int_0^\tau d\tau_1 e^{i\omega\tau_1} f^*(\tau_1), \quad 2\mathrm{Re}\,\delta = |\beta|^2 \tag{4.7}$$

Let us consider the case where the quantized field is initially in the vacuum state $|0\rangle$. Applying the evolution operator (4.2) to this state, we obtain

$$|\psi(t)\rangle = U(t)|0\rangle = e^{-\delta(t)} \sum_n \frac{\left(\beta(t)e^{-i\omega t}\right)^n}{\sqrt{n!}} |n\rangle = e^{-i\mathrm{Im}\,\delta(t)} |\beta(t)e^{-i\omega t}\rangle \tag{4.8}$$

which means that classical pumping generates a coherent state $|\beta(t)e^{-i\omega t}\rangle$ from the vacuum. Since, in the coherent state, $|\beta\rangle$,

$$\langle\hat{p}\rangle = \sqrt{2\omega}\,\mathrm{Im}\,\beta(t), \quad \langle\hat{q}\rangle = \sqrt{\frac{2}{\omega}}\,\mathrm{Re}\,\beta(t) \tag{4.9}$$

we obtain

$$E_{\mathrm{cl}} = \frac{\langle p\rangle^2}{2} + \frac{\omega^2\langle q\rangle^2}{2} = \omega|\beta(t)|^2 \tag{4.10}$$

which has the interpretation of the 'classical' energy transferred to the quantized field from the pumping field. It is interesting to note that the probability for the field to remain in the vacuum state at time t is expressed in terms of the classical energy E_{cl} as [33]

$$|\langle 0|\psi(t)\rangle|^2 = \exp\left[-|\beta(t)|^2\right] = \exp\left[-E_{\mathrm{cl}}(t)\right] \tag{4.11}$$

4.2
Linear Parametric Amplifier

In the previous section, we studied the excitation of a single quantum field mode by a classical force. In this section, we analyze what happens when the classical pumping field excites two different modes of the quantum field, having frequencies ω_1 and ω_2, respectively, inside a cavity filled with a nonlinear medium, thus allowing frequency conversion to occur. We suppose, for simplicity, that the interaction between the modes is described by the following effective Hamiltonian:

$$H = \omega_1 a^\dagger a + \omega_2 b^\dagger b + g\left(a^\dagger b^\dagger e^{-i\omega t} + ab e^{i\omega t}\right) \tag{4.12}$$

where the frequency of the pumping field satisfies the resonance condition $\omega = \omega_1 + \omega_2$. It is worth noting that the Hamiltonian (4.12) is the classical limit, in the sense that $\langle c^\dagger c\rangle \to \infty$, of the following three-photon Hamiltonian

$$H = \omega_1 a^\dagger a + \omega_2 b^\dagger b + \omega c^\dagger c + g\left(a^\dagger b^\dagger c + abc^\dagger\right) \tag{4.13}$$

A specific case of the above Hamiltonian is discussed in Chapter 6.

First, we note that the Hamiltonian (4.12) admits the following integral of motion:

$$N = a^\dagger a - b^\dagger b, \quad [H, N] = 0 \tag{4.14}$$

Clearly, this is just the difference between the number of excitations in the modes "a" and "b". The Hamiltonian (4.12) can be rewritten in terms of generators of the $su(1,1)$ algebra (see Appendix 11.3),

$$K_0 = \frac{1}{2}\left(a^\dagger a + b^\dagger b + 1\right), \quad K_+ = a^\dagger b^\dagger, \quad K_- = ab \tag{4.15}$$

in the following way:

$$H = \omega K_0 + f(t) K_+ + f^*(t) K_- + c \tag{4.16}$$

where

$$\omega = \omega_1 + \omega_2, \quad f(t) = g\mathrm{e}^{-\mathrm{i}\omega t}, \quad c = \frac{1}{2}(\omega_1 - \omega_2)N - \frac{\omega}{2} \tag{4.17}$$

Here, the term c commutes with the generators of the $su(1,1)$ algebra and can be neglected as it leads only to a global phase in the evolution. The Hilbert space for this model is spanned by basis states of the form

$$|n_a, m_b\rangle = |n\rangle_a \otimes |m\rangle_b, \quad n_a - m_b = N = \text{const} \tag{4.18}$$

where $|n\rangle_a$ and $|m\rangle_b$ are the number states corresponding to the "a" and "b" modes. It can be shown that the value of the Casimir operator for the $su(1,1)$ algebra,

$$C_2 = K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+), \quad [C_2, K_{\pm,0}] = 0 \tag{4.19}$$

in this space, is equal to $C_2 = k\left(k-1\right)$, where $k = 1/2\,(1 + |N|)$. We look for the evolution operator in the disentangled form (Gauss decomposition),

$$U(t) = \mathrm{e}^{\alpha(t)K_+}\mathrm{e}^{\beta(t)K_0}\mathrm{e}^{\gamma(t)K_-} \tag{4.20}$$

To determine the parameters in this expansion, we substitute Equation 4.20 into the Schrödinger equation, $\mathrm{i}\dot{U} = HU$, and make use of the transformations

$$\begin{aligned} \mathrm{e}^{\alpha K_+}K_0\mathrm{e}^{-\alpha K_+} &= K_0 - \alpha K_+ \\ \mathrm{e}^{\beta K_0}K_-\mathrm{e}^{-\beta K_0} &= K_-\mathrm{e}^{-\beta} \\ \mathrm{e}^{\gamma K_+}K_-\mathrm{e}^{-\gamma K_+} &= K_- - 2\gamma K_0 + \gamma^2 K_+ \end{aligned} \tag{4.21}$$

to obtain the following equality:

$$\begin{aligned} &\dot{\alpha}K_+ + \dot{\beta}(K_0 - \alpha K_+) + \dot{\gamma}\mathrm{e}^{-\beta}\left(K_- - 2\alpha K_0 + \alpha^2 K_+\right) \\ &\quad = -\mathrm{i}\left(\omega K_0 + f(t)K_+ + f^*(t)K_-\right) \end{aligned} \tag{4.22}$$

As the generators are linearly independent, we can equate their respective coefficients and arrive at the system of ordinary differential equations:

$$\dot{\alpha} - \alpha\dot{\beta} + \alpha^2 e^{-\beta}\dot{\gamma} = -if, \quad \dot{\beta} + 2\alpha e^{-\beta}\dot{\gamma} = -i\omega, \quad e^{-\beta}\dot{\gamma} = -if^* \tag{4.23}$$

subject to the initial conditions $\alpha(0) = \beta(0) = \gamma(0) = 0$. The solution to this system is given by

$$\alpha(t) = -ie^{-i\omega t}\tanh gt, \quad \beta(t) = -i\omega t - 2\ln(\cosh gt), \quad \gamma(t) = -i\tanh gt \tag{4.24}$$

Using the form (4.20), we observe that the initial vacuum state of both modes $|0_a, 0_b\rangle = |0_a\rangle|0_b\rangle$ evolves into a completely correlated state,

$$U(t)|0_a, 0_b\rangle = \left(\cosh gt\right)^{-1} e^{-i\omega t/2} \sum_n (-i)^n e^{-i\omega nt} \left(\tanh gt\right)^n |n\rangle_a |n\rangle_b \tag{4.25}$$

In other words, the photons in the modes "a" and "b" are created in pairs.

Using the evolution operator in the form (4.20), we can find the photon-number operators in each mode in the Heisenberg representation,

$$\hat{n}_a(t) = U^\dagger(t) a^\dagger a U(t), \quad \hat{n}_b(t) = U^\dagger(t) b^\dagger b U(t) \tag{4.26}$$

Indeed, the operators $a^\dagger a$ and $b^\dagger b$ can be expressed in terms of generators of the $su(1, 1)$ algebra,

$$a^\dagger a = K_0 + \frac{1}{2}(N - 1), \quad b^\dagger b = K_0 - \frac{1}{2}(N + 1) \tag{4.27}$$

Considering the formulae (4.21), we obtain

$$\hat{n}_{a,b}(t) = K_0 \cosh 2gt - \frac{i}{2}\sinh 2gt\,(K_+ - K_-) - \frac{1}{2}(1 \mp N) \tag{4.28}$$

so that,

$$\hat{n}_a(t) = \hat{n}_a(0)\cosh 2gt + (1 - N)\sinh^2 gt - \frac{i}{2}(a^\dagger b^\dagger - ab)\sinh 2gt$$

$$\hat{n}_b(t) = \hat{n}_b(0)\cosh 2gt + (1 + N)\sinh^2 gt - \frac{i}{2}(a^\dagger b^\dagger - ab)\sinh 2gt$$

This implies that, for example, starting from the vacuum state of both modes $|0_a, 0_b\rangle$, the number of excitations in each mode increases exponentially,

$$\langle 0_a, 0_b|\hat{n}_{a,b}(t)|0_a, 0_b\rangle = \sinh^2 gt \tag{4.29}$$

The algebraic structure of the solution is preserved also in the case of the degenerate parametric amplifier, which describes an excitation of a single quantum field mode of frequency ω in a nonlinear medium under pumping by a classical field with the frequency 2ω. The Hamiltonian (4.12) of this model is written as follows:

$$H = \omega a^\dagger a + \frac{g}{2}\left(a^{\dagger 2} e^{-2i\omega t} + a^2 e^{2i\omega t}\right) \tag{4.30}$$

It can also be represented in terms of the generators of the $su(1,1)$ algebra:

$$H = 2\omega K_0 + f(t)K_+ + f^*(t)K_- - \frac{1}{2}\omega I, \quad f(t) = g e^{-2i\omega t} \tag{4.31}$$

where

$$K_0 = \frac{1}{2}\left(a^\dagger a + \frac{1}{2}\right), \quad K_+ = \frac{1}{2}a^{\dagger 2}, \quad K_- = \frac{1}{2}a^2 \tag{4.32}$$

The evolution operator for the Hamiltonian (4.31) has the same form (4.20), with the parameters $\alpha(t), \beta(t), \gamma(t)$ defined by Equation 4.21, provided one makes the replacement $\omega \to 2\omega$. However, the representation space for this model is different from the case of the nondegenerate amplifier. The whole representation space, which in this case is the space of a single harmonic oscillator, is divided into two invariant subspaces, the first including the even states, $\{|2n\rangle, n = 0, 1\ldots\}$ and the second, the odd states, $\{|2n+1\rangle, n = 0, 1\ldots\}$. The Casimir operator takes the value $C_2 = -3/16 = k(k-1)$, where $k = 1/4$ in the subspace with an even number of photons and $k = 3/4$ in the subspace with an odd number of photons.

It can easily be seen that an initial coherent state of the quantum field evolves into a squeezed state. In particular, the vacuum state evolves to the 'squeezed vacuum' (3.115)

$$|0,\eta\rangle = \left(\cosh r\right)^{-1/2} \sum_{n=0}^{\infty} \left[-\frac{1}{2} e^{i\theta} \tanh r\right]^n \frac{\sqrt{(2n)!}}{n!} |2n\rangle \tag{4.33}$$

where $r = gt$ and $\theta = -2\omega t + \pi/2$. The photon-number operator, which also grows exponentially with time, can be found from Equation 4.28

$$\hat{n}(t) = \hat{n}(0)\cosh 2gt + \sinh^2 gt - \frac{i}{2}(a^{\dagger 2} - a^2)\sinh 2gt \tag{4.34}$$

4.3
Evolution in the Kerr Medium

Let us consider here a simple model of a 'self-interacting' quantum field that describes the electromagnetic field propagation in an optical medium with a refraction index depending on the field intensity (Kerr medium) [34, 35]. In the case of a single field mode, the Hamiltonian has the form

$$H = \omega\hat{n} + \chi\hat{n}^2 \tag{4.35}$$

First, we find the field operators in the Heisenberg representation. Using the commutation relation $f(\hat{n})a^\dagger = a^\dagger f(\hat{n}+1)$, we obtain,

$$e^{i\tau\hat{n}^2} a^\dagger e^{-i\tau\hat{n}^2} = e^{i\tau[(\hat{n}+1)^2 - \hat{n}^2]} a^\dagger = e^{i\tau(2\hat{n}+1)} a^\dagger \tag{4.36}$$

Thus, the operator $a^\dagger(t) = \exp(\mathrm{i}tH)a^\dagger \exp(-\mathrm{i}tH)$ has the form

$$a^\dagger(t) = \mathrm{e}^{\mathrm{i}(\omega+\chi)t}\mathrm{e}^{\mathrm{i}2\chi t\hat{n}}a^\dagger, \quad \text{and} \quad a(t) = a\mathrm{e}^{-\mathrm{i}2\chi t\hat{n}}\mathrm{e}^{-\mathrm{i}(\omega+\chi)t} \tag{4.37}$$

This example of evolution in the Kerr medium demonstrates the difference between quantum and classical dynamics in a spectacular way [35]. We suppose that the initial state of the classical system is given by a Gaussian probability distribution (in the sense of classical statistical mechanics) in the phase space. The classical evolution deforms the initial distribution (a round hump) into a sort of spiral. (The distribution parts that are farthest away from the origin correspond to the highest intensities and move faster.) Quantum evolution corresponds to the classical evolution only for very short times. For longer times ($\chi t \sim 1/\sqrt{\bar{n}}$), the phase fluctuations exceed 2π and quantum evolution leads to self-interference in the phase space, while classical evolution leads only to a simple intensity redistribution. To be more exact, in the case of quantum evolution, at certain times, well-pronounced standing waves are created in the phase space; known as *Schrödinger cats* these are superpositions of the macroscopically distinguishable quantum states [36, 37].

Indeed, one can show that the evolution operator

$$U(t) = \exp\left[-\mathrm{i}t\left(\omega\hat{n} + \chi\hat{n}^2\right)\right] \tag{4.38}$$

leads to the creation of M copies of the initial state (with different phases) at times $\chi t = L\pi/M$ (where L and M are integers). In particular, if $M = 1$ (or $\chi t = L\pi$), the initial state is reconstructed.

To quantitatively describe this phenomenon, we apply the finite Fourier transform to the evolution operator (4.38) at $\chi t = \pi L/M$. For simplicity, we consider the case $L = 1$; thus, for even and odd values of M, we obtain

$$U_M(\hat{n}) = \mathrm{e}^{-\mathrm{i}\pi\hat{n}^2/M} = \sum_{q=0}^{M-1} f_q^M \mathrm{e}^{-2\mathrm{i}\pi q\hat{n}/M}, \quad M \text{ even} \tag{4.39}$$

$$U_M(\hat{n}) = \mathrm{e}^{-\mathrm{i}\pi\hat{n}(\hat{n}-1)/M} = \sum_{q=0}^{M-1} f_q^M \mathrm{e}^{-2\mathrm{i}\pi q\hat{n}/M}, \quad M \text{ odd} \tag{4.40}$$

The amplitudes of different components of the "cat" states are found using the inverse Fourier transform:

$$f_q^M = \frac{1}{M}\sum_{n=0}^{M-1} \mathrm{e}^{2\mathrm{i}\pi qn/M}\, U_M(n) \tag{4.41}$$

These coefficients are given as [38]

$$f_q^M = \frac{1}{\sqrt{M}}\exp\left[\frac{\mathrm{i}\pi q^2}{M}\right]\exp\left[\frac{-\mathrm{i}\pi}{4}\right], \quad M \text{ even}$$

$$f_q^M = \frac{1}{\sqrt{M}}\exp\left[\frac{\mathrm{i}\pi(q-1/2)^2}{M}\right]\exp\left[\frac{-\mathrm{i}\pi}{4}\right], \quad M \text{ odd} \tag{4.42}$$

Applying the evolution operator in the form of Equations 4.39 and 4.40 to the coherent initial state $|\alpha\rangle$ we find the wave function for the cat state can be expressed as a superposition of coherent states along a circle of radius $|\alpha|$ in the phase space of the field:

$$U_M(\hat{n})|\alpha\rangle = \sum_{q=0}^{M-1} f_q^M \mathrm{e}^{-2\mathrm{i}\pi q\hat{n}/M}|\alpha\rangle = \sum_{q=0}^{M-1} f_q^M |\alpha \mathrm{e}^{-2\mathrm{i}\pi q/M}\rangle \tag{4.43}$$

In particular, for $\chi t = \pi/2$, the density matrix becomes $\rho_a(\pi/2\lambda) = |\psi_c\rangle_a\, {}_a\langle\psi_c|$, where $|\psi_c\rangle_a$ is a superposition of two coherent states with opposite phases:

$$|\psi_c\rangle_a = \frac{1}{\sqrt{2}}\left[\mathrm{e}^{\mathrm{i}\pi/4}|\alpha\rangle_a + \mathrm{e}^{-\mathrm{i}\pi/4}|-\alpha\rangle_a\right] \tag{4.44}$$

4.4
Second Harmonic Generation in the Dispersive Limit

Second harmonic generation is usually described by the following model Hamiltonian

$$H = H_0 + H_{\text{int}} = \omega_a a^\dagger a + \omega_b b^\dagger b + g(a^2 b^\dagger + a^{\dagger 2} b) \tag{4.45}$$

where a ($a^\dagger$) and b ($b^\dagger$) are the annihilation (creation) operators of the fundamental mode (of frequency ω_a) and of the second harmonic mode (of frequency ω_b), respectively. When perfect matching conditions are satisfied, we have the relation $\omega_b = 2\omega_a$. The constant g describes phenomenologically the coupling between the modes. It can always be chosen as real.
If we note that the Hamiltonian (4.45) admits the constant of motion

$$N = a^\dagger a + 2b^\dagger b \tag{4.46}$$

then the Hamiltonian can be rewritten in the following form:

$$\begin{aligned} H_0 &= \frac{\omega_b + \omega_a}{3} N \\ H_{\text{int}} &= \frac{\Delta}{3}(b^\dagger b - a^\dagger a) + g(a^2 b^\dagger + a^{\dagger 2} b) \end{aligned} \tag{4.47}$$

where $\Delta = \omega_b - 2\omega_a$ is the detuning.

Since H_0 determines the total energy stored in both modes, which is conserved, $[H_0, H_{\text{int}}] = 0$, we can factor out $\exp(-\mathrm{i}H_0 t)$ from the evolution operator and drop it altogether.

The corresponding classical problem admits an exact solution, demonstrating the possibility of the complete energy transfer into the second harmonic mode [39]. Unfortunately, the description of corresponding quantum dynamics is a complicated problem, though different algebraic, semiclassical, and numerical approaches have been developed [40–42]. In particular, it was numerically observed [43] that, in the exact resonance case with the fundamental mode initially in a highly excited coherent state, the Schrödinger cat type states (superpositions of macroscopically distinguishable states) are formed (and later destroyed) for long times.

In this section, we are interested in the dispersive limit of this model, when

$$|\Delta| \gg g(\overline{n}_a + 1)(\overline{n}_b + 1) \tag{4.48}$$

where $\overline{n}_a$ and $\overline{n}_b$ denote the average photon numbers in the modes a and b, respectively. Then, using the Lie transformation method (see Appendix 11.4), we apply a unitary transformation to the interaction Hamiltonian (4.45)

$$H_{\text{eff}} = U H_{\text{int}} U^{\dagger} \tag{4.49}$$

where

$$U = \exp\left[\lambda(a^2 b^{\dagger} - a^{\dagger 2} b)\right] \tag{4.50}$$

with $\lambda = g^2/\Delta \ll 1$. By expanding Equation 4.49 in a power series and keeping terms up to the order $(g/\Delta)^2$, we get

$$H_{\text{eff}} = \frac{\Delta}{3}(b^{\dagger}b - a^{\dagger}a) - \lambda\left[4b^{\dagger}ba^{\dagger}a - (a^{\dagger}a)^2\right] \tag{4.51}$$

The effective Hamiltonian (4.51) describes the dispersive evolution of the fields, but the essential point is that it is diagonal, which implies that there is no population transfer between the modes (as expected in the far-off resonant limit). The first term in Equation 4.51 does not affect the dynamics and just leads to rapid oscillations of the wave function.

Now, it is easy to find the evolution of the density matrix of the fundamental mode. When the second harmonic mode is initially in the vacuum state, the two terms containing $b^{\dagger}b$ in Equation 4.51 do not contribute in the evolution. In addition, the linear term in $a^{\dagger}a$ leads just to a c-number phase shift and can be omitted. Then, the effective Hamiltonian reduces to

$$H_K = \lambda(a^{\dagger}a)^2 \tag{4.52}$$

which is nothing but the interaction Hamiltonian that governs the state evolution of a single-mode field a in a Kerr medium (see Section 3, Chapter 4).

When the initial state of the second harmonic mode is a superposition of number states, the dynamics of the fundamental mode is drastically affected. For simplicity, we suppose that both modes are initially coherent states, namely, $|\alpha\rangle_a \otimes |\beta\rangle_b$. Then, we obtain the following expression for the density matrix of the fundamental mode:

$$\rho_a(t) = e^{-|\alpha|^2} \sum_{n,m}^{\infty} e^{-it\lambda(n^2 - m^2)} \exp\left\{|\beta|^2\left[1 - e^{-4i\lambda t(n-m)}\right]\right\} \frac{\alpha^n \alpha^{*m}}{\sqrt{n!m!}} |n\rangle_{aa}\langle m| \tag{4.53}$$

Clearly, due to the presence of the term $b^{\dagger}ba^{\dagger}a$ in Equation 4.51, the modes a and b get entangled in the course of evolution. However, when $\lambda t = \pi/2$, they become decoupled and two copies of the initial coherent state in the fundamental mode are created. Indeed, the density matrix of the total system at this instant of time is

$$\rho(\pi/2\lambda) = |\beta\rangle_{bb}\langle\beta| \otimes |\psi_c\rangle_{aa}\langle\psi_c| \tag{4.54}$$

and a two-component cat state can be produced in this case. In contrast to this, the initial coherent state reappears when $\lambda t = \pi$, i.e. it gets reconstructed.

The effective Hamiltonian (4.51) has the virtue of simplifying calculations that otherwise would be rather involved. For example, the squeezing properties of the fundamental mode are also affected by the initial state of the second harmonic mode. For initial coherent states in both modes $|\alpha\rangle_a \otimes |\beta\rangle_b$, and, assuming, for simplicity, that α and β are real, we easily get the fluctuations $\sigma_q = \langle q^2\rangle - \langle q\rangle^2$ of the quadrature $q = (a + a^\dagger)/\sqrt{2}$.

$$\begin{aligned}\sigma_q(t) = {}& \frac{1}{2} + \alpha^2 \left(e^{-8z^2T^2} - e^{-4z^2T^2}\right)\cos\Omega T \\ & -2\alpha^2T^2\left(4e^{-8z^2T^2} - e^{-4z^2T^2}\right)\cos\Omega T \\ & -2\alpha^2T\left(2e^{-8z^2T^2} - e^{-4z^2T^2}\right)\sin\Omega T + \alpha^2\left(1 - e^{-4z^2T^2}\right)\end{aligned}$$

where $T = \lambda t \ll 1$, $z^2 = \alpha^2 + 4\beta^2$, and $\Omega = 4\alpha^2 - 8\beta^2$. Then, for sufficiently large values of β^2, squeezing in the fundamental mode disappears.

4.5
Raman Dispersion

Now let us consider the pairwise interactions of three separate field modes. The Hamiltonian of the system has the form

$$H = \omega_a a^\dagger a + \omega_b b^\dagger b + \omega_c c^\dagger c + g_1\left(c^\dagger a + a^\dagger c\right) + g_2\left(c^\dagger b + b^\dagger c\right) \tag{4.55}$$

with the interaction constants g_1 and g_2. With the observation that this Hamiltonian admits two integrals of motions,

$$N_a = a^\dagger a + c^\dagger c, \quad N_b = b^\dagger b + c^\dagger c \tag{4.56}$$

we can rewrite Equation 4.55 in the following form:

$$\begin{aligned} H &= \omega_a N_a + \omega_b N_b + H_{\text{int}} \\ H_{\text{int}} &= \delta c^\dagger c + g_1\left(c^\dagger a + a^\dagger c\right) + g_2\left(c^\dagger b + b^\dagger c\right)\end{aligned} \tag{4.57}$$

where $\delta = \omega_c - \omega_a - \omega_b$. It is convenient to express this Hamiltonian as

$$H_{\text{int}} = \delta\, c^\dagger c + \sqrt{g_1^2 + g_2^2}\left(c^\dagger d + c d^\dagger\right) \tag{4.58}$$

where the operators

$$d = \frac{g_1}{\sqrt{g_1^2+g_2^2}}a + \frac{g_2}{\sqrt{g_1^2+g_2^2}}b, \quad d^\dagger = \frac{g_1}{\sqrt{g_1^2+g_2^2}}a^\dagger + \frac{g_2}{\sqrt{g_1^2+g_2^2}}b^\dagger$$

satisfy the $H(1)$ commutation relations, $\left[d, d^\dagger\right] = I$. The Hamiltonian (4.58) is a linear form on the $su(2)$ algebra generators,

$$H_{\text{int}} = \frac{\delta}{2}N + \frac{\delta}{2}S_z + 2\sqrt{g_1^2 + g_2^2}\,S_x, \quad [N, H_{\text{int}}] = 0$$

where $N = d^\dagger d + c^\dagger c$, $S_z = (c^\dagger c - d^\dagger d)/2$ and $S_x = (c^\dagger d + c d^\dagger)/2$, so that both spectrum and the evolution operator can be easily found (see Appendix 11.3).

Nevertheless, we focus on the case when the mode c is far from resonance with the modes a and b, so that $\delta \gg g_1\sqrt{(\overline{n}_a + 1)\,(\overline{n}_c + 1)}$ and $\delta \gg g_2\sqrt{(\overline{n}_b + 1)\,(\overline{n}_c + 1)}$, when the mode c can be adiabatically eliminated.

Following, once more, the prescriptions of Lie transformation methods, (see Appendix 11.4) we are able to remove the terms that describe photon exchange between the modes $a \leftrightarrow c$ and $b \leftrightarrow c$ through the application of small rotations.

First, we apply the unitary transformation U_1 given by

$$U_1 = \exp\left[\epsilon_1\left(a^\dagger c - c^\dagger a\right)\right] \tag{4.59}$$

where $\epsilon_1 = g_1/\delta \ll 1$, obtaining

$$\begin{aligned} H_1 = U_1 H_{\text{int}} U_1^\dagger &\approx \delta c^\dagger c + \frac{g_1^2}{\delta}\left(c^\dagger c - a^\dagger a\right) + \\ &+ g_2\left(c^\dagger b + b^\dagger c\right) - \frac{g_1 g_2}{\delta}\left(b^\dagger a + a^\dagger b\right) + O\left(\delta^{-2}\right) \end{aligned} \tag{4.60}$$

In the Hamiltonian (4.60), we still have interaction terms between the modes $b \leftrightarrow c$. To eliminate this term, we apply a second rotation U_2

$$U_2 = \exp\left[\epsilon_1\left(b^\dagger c - c^\dagger b\right)\right], \quad \epsilon_2 = \frac{g_2}{\delta} \ll 1 \tag{4.61}$$

Finally, arriving at the following effective Hamiltonian:

$$\begin{aligned} H_{\text{eff}} = U_2^\dagger H_1 U_2 &\approx \delta c^\dagger c + \frac{g_1^2}{\delta}\left(c^\dagger c - a^\dagger a\right) \\ &+ \frac{g_2^2}{\delta}\left(c^\dagger c - b^\dagger b\right) - \frac{2 g_1 g_2}{\delta}\left(b^\dagger a + a^\dagger b\right) + O\left(\delta^{-2}\right) \end{aligned} \tag{4.62}$$

The Hamiltonian (4.62) describes effective transitions between the modes a and b and contains no terms that allow the exchange of photons with the mode c. Thus, the operator $c^\dagger c$ is approximately conserved,

$$\left[H_{\text{eff}}, c^\dagger c\right] \approx 0 \tag{4.63}$$

and, if, initially, there were no photons in the mode c, this mode never gets excited; in this case, the effective Hamiltonian takes the form

$$H_{\text{eff}} \approx \frac{g_1^2}{\delta} a^\dagger a + \frac{g_2^2}{\delta} b^\dagger b + \frac{2 g_1 g_2}{\delta}\left(b^\dagger a + a^\dagger b\right) \tag{4.64}$$

where we have changed the overall sign of the Hamiltonian. This means that the three-mode system (4.55) can be treated as an effective two-mode system with linear interaction among the modes and with the effective coupling constant $2g_1g_2/\delta$. The two modes are in resonance if $g_1 = g_2$.

4.6
Problems

4.1 Prove that during the evolution in the Kerr medium, $H = \chi(a^\dagger a)^2$, a coherent initial state $|\alpha\rangle$ is transformed into a squeezed state. Find the maximum squeezing direction and the degree of squeezing if the initial state is highly excited, $\alpha \gg 1$.

4.2 Prove the following expression for the average value for the initial coherent state

$$\langle\psi(t)|a^{\dagger n}a^{n+m}|\psi(t)\rangle = \alpha^m \overline{n}^n \mathrm{e}^{-\overline{n}(1-\cos 2m\chi t)} \times \exp\left[-\mathrm{i}\chi t\left(m^2+2mn\right) - \mathrm{i}\overline{n}\sin 2m\chi t\right] \quad (4.65)$$

where $|\psi(t)\rangle = U(t)|\alpha\rangle$, and U(t) is the evolution operator in the Kerr medium, Equation 4.38.

4.3 The Hamiltonian that describes the interaction between two modes of the quantum field has a form, $H = \omega a^\dagger a + g(ab^\dagger + ba^\dagger)$. Determine the evolution of the initial state $|\psi(0)\rangle = |\alpha\rangle_a|0\rangle_b$. Hint: write the evolution operator in the form:

$$U(t) = \mathrm{e}^{-i\omega t/2(a^\dagger a+b^\dagger b)} \exp\left(\chi(t)ab^\dagger\right) \exp\left(\nu(t)\left(a^\dagger a - b^\dagger b\right)\right) \times \exp\left(\mu(t)ba^\dagger\right) \quad (4.66)$$

Discuss the possibility of generating squeezed states in one of the modes.

4.4 Prove that the effective Hamiltonian describing the *three-wave mixing* process,

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + \omega_3 a_3^\dagger a_3 + g(a_1 a_2 a_3^\dagger + a_1^\dagger a_2^\dagger a_3) \quad (4.67)$$

in the dispersive limit, $|\Delta| \gg g\sqrt{(\overline{n}_1+1)(\overline{n}_2+1)(\overline{n}_3+1)}$, where $\Delta = \omega_1 + \omega_2 - \omega_3$ and $\overline{n}_{1,2,3}$ are the average number of photons in the corresponding modes, has the form

$$H_{\text{eff}} = \Delta a_1^\dagger a_1 - \frac{g^2}{\Delta}\left[a_1^\dagger a_1(a_3^\dagger a_3 - a_2^\dagger a_2) + a_2 a_2^\dagger a_3^\dagger a_3\right] \quad (4.68)$$

Hint. Take into account that the Hamiltonian (4.67) admits two integrals of motion, $N_1 = a_1^\dagger a_1 + a_2^\dagger a_2 + 2a_3^\dagger a_3$, and $N_2 = a_1^\dagger a_1 - a_2^\dagger a_2$.

4.5 Prove that the effective Hamiltonian describing *kth harmonic generation*,

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + g(a_1^k a_2^{\dagger n} + a_1^{\dagger k} a_2^n) \tag{4.69}$$

in the dispersive limit, $\Delta \gg g(\overline{n}_1 + 1)^{k/2}(\overline{n}_2 + 1)^{n/2}$, where $\Delta = n\omega_2 - k\omega_1$ and $\overline{n}_{1,2}$ denote the average photon numbers in modes 1 and 2, has the form

$$\begin{aligned} H_{\text{eff}} = {} & \frac{n\omega_2 - k\omega_1}{n+k}(a_2^\dagger a_2 - a_1^\dagger a_1) \\ & + \frac{g^2}{\Delta}\left[\phi_n(a_2^\dagger a_2)\ \psi_{k-1}(a_1^\dagger a_1) - \phi_k(a_1^\dagger a_1)\ \psi_{n-1}(a_2^\dagger a_2)\right] \end{aligned} \tag{4.70}$$

where $\phi_k(m) = m^{(k)} = m(m-1)\cdots(m-k+1)$ and $\psi_{k-1}(m) = \phi_k(m+k) - \phi_k(m)$.

Hint: Take into account that the Hamiltonian (4.69) admits the integral of motion $N = (ka_1^\dagger a_1 + na_2^\dagger a_2)/(n+k)$.

5
The Jaynes–Cummings Model

5.1
The Interaction Hamiltonian

We recall (see Section 3.1) that the operator of a free electromagnetic field in a perfect cavity (in the Coulomb gauge) has the form

$$\mathcal{E} = \mathrm{i}\sum_k \boldsymbol{\epsilon}_k \sqrt{\frac{2\pi\hbar\omega_k}{V}} \left(a_k\,\mathrm{e}^{\mathrm{i}\mathbf{kr}} - a_k^\dagger\,\mathrm{e}^{-\mathrm{i}\mathbf{kr}}\right)$$

where $a_k, a_k^\dagger$ are the annihilation and creation operators of photons with the wave vector $\mathbf{k}$, frequency ω_k, and polarization $\boldsymbol{\epsilon}_k$. They satisfy the common boson commutation relations $[a_k, a_l^\dagger] = \delta_{kl}$; V stands for the cavity volume. The free field Hamiltonian has the form

$$H_\mathrm{f} = \sum_k \hbar\omega_k \left(a_k^\dagger a_k + \frac{1}{2}\right)$$

Let us consider a collection of identical two-level atoms with the upper energy level $|1\rangle$ and the lower energy level $|0\rangle$, so that the free atomic Hamiltonian has the form (see Section 1.1)

$$H_\mathrm{a} = \frac{\hbar\omega_\mathrm{a}}{2} \sum_j \sigma_{z_j}$$

where $\sigma_{z_j} = |1_j\rangle\langle 1_j| - |0_j\rangle\langle 0_j|$ and $\hbar\omega_\mathrm{a}$ is the energy separation between two levels. The electric dipole interaction of the cavity field with the atomic system is described by the Hamiltonian

$$V_\mathrm{fa} = -\sum_j \left(\mathbf{d}_j \cdot \mathcal{E}_j\right)$$

where $\mathcal{E}_j$ is the electric field at the point $\mathbf{r}_j$ that corresponds to the position of the jth atom and $\mathbf{d}_j$ is the electric dipole operator of this atom, given by Equation 1.6, which can now conveniently be rewritten in the following form:

$$\mathbf{d}_j = \mathbf{d}_j \sigma_{+_j} + \mathbf{d}_j^* \sigma_{-j}$$

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Here $\sigma_{+j} = |1_j\rangle\langle 0_j|$ and $\sigma_{-j} = |0_j\rangle\langle 1_j|$ are the atomic transition operators and $\mathbf{d}_j = e\langle 1_j|\mathbf{r}\,|0_j\rangle$ is the matrix element of the electric dipole momentum (which are assumed to be the same for all atoms).

If the dimensions of the atomic system are small in comparison with the wavelength of the field (but the wave functions of different atoms do not overlap, to avoid the dipole–dipole interaction), one can apply the so-called dipole approximation: $\mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{r}_j} \approx 1$. Then the interaction Hamiltonian can be rewritten in the following form:

$$V_{\mathrm{fa}} = \sum_k \left(g_k S_+ a_k + g_k^* S_- a_k^\dagger - g_k S_+ a_k^\dagger - g_k^* S_- a_k \right) \tag{5.1}$$

Here the interaction constant is given by

$$g_k = -\mathrm{i} d_k \sqrt{\frac{\pi \hbar \omega_k}{2V}}$$

where $d_k = (\mathbf{d} \cdot \boldsymbol{\epsilon}_k)$ is a projection of the atomic dipole on the polarization direction of the k-th field mode. The atomic dipole matrix element can be chosen as purely imaginary without loss of generality; it then corresponds to the positive coupling constants g_k. In Equation 5.1 we have introduced collective atomic operators (see Section 1.2)

$$S_z = \frac{1}{2} \sum_j \sigma_{zj}, \quad S_\pm = \sum_j \sigma_{\pm j}$$

Let us assume now that only one cavity mode of frequency ω is important and all of the other field modes, which can exist inside this cavity, are far from resonance with the atomic transition under study. We arrive at a single mode Hamiltonian

$$H = H_0 + g\left(S_+ a + S_- a^\dagger - S_+ a^\dagger - S_- a\right) \tag{5.2}$$

$$H_0 = \hbar\omega_{\mathrm{a}} S_z + \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$$

The interaction Hamiltonian (5.2) can be naturally divided into two parts:

$$V_r = g\left(S_- a^\dagger + a S_+\right), \quad V_a = g\left(a^\dagger S_+ + a S_-\right)$$

The term $a^\dagger S_-$ corresponds to the atomic transition from the excited to the ground state with the emission of a photon; the term aS_+ corresponds to the excitation of the atom with the absorption of a photon. Hence, the part V_r of the interaction Hamiltonian conserves the number of excitations in the system. Correspondingly, the excitation-number operator

$$\hat{N} = S_z + a^\dagger a$$

commutes with V_r:

$$\left[V_r, \hat{N} \right] = 0$$

On the other hand, the part V_a of the interaction Hamiltonian does not conserve the excitation number; the term $a^\dagger S_+$ describes the atomic excitation with the simultaneous creation of a photon, while the term aS_- corresponds to the inverse process. In the interaction picture, the Hamiltonian (5.2) has the form

$$\begin{aligned} H_I = \mathrm{e}^{(\mathrm{i}/\hbar)H_0 t} H \, \mathrm{e}^{-(\mathrm{i}/\hbar)H_0 t} &= g\left(a^\dagger S_- \, \mathrm{e}^{\mathrm{i}(\omega-\omega_\mathrm{a})\,t} + aS_+ \, \mathrm{e}^{-\mathrm{i}(\omega-\omega_\mathrm{a})\,t}\right) \\ &- g\left(a^\dagger S_+ \, \mathrm{e}^{\mathrm{i}(\omega+\omega_\mathrm{a})\,t} + aS_- \, \mathrm{e}^{-\mathrm{i}(\omega+\omega_\mathrm{a})\,t}\right) \end{aligned}$$

In the case when $\omega \sim \omega_\mathrm{a}$, the first part of the above expression is almost independent of time, while the second part rapidly oscillates and cannot significantly affect the slow evolution. We may, therefore, neglect the rapidly oscillating terms, making the rotating wave approximation (RWA). In the Schrödinger picture, we neglect the term V_a (the 'antirotating part') and the Hamiltonian takes the form

$$H = H_0 + V_r$$

This explains the subscript in the notation V_r. It is clear that under the RWA, the excitation number becomes the integral of motion

$$\mathrm{i}\hbar\dot{\hat{N}} = \left[\hat{N}, H\right] = \left[\hat{N}, V\right] = 0$$

In the case of a single atom, the Hamiltonian (5.2) describes the Jaynes–Cummings model (JCM) [44] (see also, [45]). The case of many atoms corresponds to the Dicke model (DM) [13]. The term V_a plays an important role in the off-resonance case, when the atomic transition frequency is different from the field frequency. For instance, this term leads to multiphoton processes when the multiphoton resonance conditions are satisfied.

5.2
The Spectrum and Wave Functions

Using the units when $\hbar = 1$, the Hamiltonian can be rewritten in the form

$$H = \omega\left(\hat{N} + \frac{1}{2}\right) + \frac{\Delta}{2}\sigma_z + g\left(a^\dagger\sigma_- + a\sigma_+\right)$$

where $\Delta = \omega_\mathrm{a} - \omega$ stands for detuning, and the excitation number operator is $\hat{N} = a^\dagger a + \sigma_z/2$. This Hamiltonian acts in the tensor product of the two-dimensional atomic Hilbert space and the infinite-dimensional field space with the basis

$$|k\rangle_a \otimes |n\rangle_f \tag{5.3}$$

where $|n\rangle_f$ is the Fock (number) state of the field, $a^\dagger a|n\rangle_f = n|n\rangle_f$ $(n = 0, 1, 2\ldots)$, and $|k\rangle_a$ is the eigenstate of the operator σ_z, $\sigma_z|k\rangle_a = (2k-1)|k\rangle_a$ $(k = 0, 1)$.

If the initial state belongs to the subspace with a given excitation number N, then the evolution is restricted to this subspace (as it follows from the conservation of the excitation number). We may introduce a different notation for the basis vectors (Equation 5.3), stressing the fact that they are eigenvectors of the excitation operator

$$|N,k\rangle = |k\rangle_a \otimes |N-k\rangle_f, \quad k=0,1,$$
$$\hat{N}|N,k\rangle = \left(N-\frac{1}{2}\right)|N,k\rangle$$

Subspaces with N excitations are two-dimensional (except for the one dimensional subspace $N=0$). One may use the vector notation

$$|N,0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad |N,1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Inside a single subspace, the Hamiltonian takes the matrix form

$$H = \omega N \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{\Delta}{2} & g\sqrt{N} \\ g\sqrt{N} & -\frac{\Delta}{2} \end{bmatrix}$$

The eigenvalues of the Hamiltonian read

$$\Lambda_{N,0} = \omega N + \Omega, \quad \Lambda_{N,1} = \omega N - \Omega, \quad \Omega = \sqrt{g^2 N + \frac{\Delta^2}{4}}$$

and the normalized eigenvectors are

$$\underline{|N,0\rangle} = \begin{bmatrix} x_N \\ y_N \end{bmatrix}, \quad \underline{|N,1\rangle} = \begin{bmatrix} -y_N \\ x_N \end{bmatrix} \tag{5.4}$$

where

$$x_N = \frac{g\sqrt{N}}{\sqrt{\left(\Omega - \frac{\Delta}{2}\right)^2 + g^2 N}}, \quad y_N = \frac{\Omega - \frac{\Delta}{2}}{\sqrt{\left(\Omega - \frac{\Delta}{2}\right)^2 + g^2 N}}$$

To determine the evolution of an arbitrary initial state $|\zeta\rangle_a \otimes |\phi\rangle_f$, we need to expand it in the dressed basis $\underline{|N,k\rangle}$. For instance, for the initial state

$$|\psi(0)\rangle = |0\rangle_a \otimes |\phi\rangle_f, \quad |\phi\rangle_f = \sum_{n=0}^{\infty} \phi_n |n\rangle_f$$

we have

$$|\psi(0)\rangle = \sum_{N=0}^{\infty} \phi_N |N,0\rangle$$

From Equations 5.4, $|N,0\rangle = y_N|\underline{N,0}\rangle + x_N|\underline{N,1}\rangle$ and

$$|\psi(0)\rangle = \sum_{N=0}^{\infty} \phi_N \left(y_N|\underline{N,0}\rangle + x_N|\underline{N,1}\rangle\right)$$

Therefore, the evolution of the initial state $|\psi(0)\rangle$ is given by

$$|\psi(t)\rangle = \sum_{N=0}^{\infty} \phi_N \left[y_N\, \mathrm{e}^{-\mathrm{i}\Lambda_{N,0}t}|\underline{N,0}\rangle + x_N\, \mathrm{e}^{-\mathrm{i}\Lambda_{N,1}t}|\underline{N,1}\rangle\right]$$

Note, that in the case of the exact resonance ($\Delta = 0$), the states $|\underline{N,k}\rangle (k = 0, 1)$ do not depend on the subspace index N:

$$|\underline{N,0}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad |\underline{N,1}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

5.3
Evolution Operator

Since the excitation-number operator commutes with the Hamiltonian, i.e. $[H, \hat{N}] = 0$, the evolution operator takes the form

$$U(t) = \mathrm{e}^{-\mathrm{i}Ht} = \mathrm{e}^{-\mathrm{i}\left(\hat{N}+1/2\right)\omega t}\,U_I, \quad U_I = \mathrm{e}^{-\mathrm{i}Vt}$$

where the excitation-number operator and interaction Hamiltonian can be written as a 2×2 matrices with respect to the atomic variables (which should not be confused with matrix notations inside a subspace with a given N that was introduced in the previous section).

$$\hat{N} = \begin{bmatrix} \hat{n} + \frac{1}{2} & 0 \\ 0 & \hat{n} - \frac{1}{2} \end{bmatrix}, \quad \hat{n} = a^{\dagger}a, \quad V = \begin{bmatrix} \frac{\Delta}{2} & ga \\ ga^{\dagger} & -\frac{\Delta}{2} \end{bmatrix}$$

To calculate the exponential of the operator V we use the Taylor series expansion:

$$V^2 = \begin{bmatrix} g^2 aa^{\dagger} + \frac{\Delta^2}{4} & 0 \\ 0 & g^2 a^{\dagger}a + \frac{\Delta^2}{4} \end{bmatrix}$$

or

$$V^2 = g^2\left(a^{\dagger}a + \frac{\sigma_z + I}{2}\right) + \frac{\Delta^2}{4} = \left[g^2\left(\hat{N} + \frac{1}{2}\right) + \frac{\Delta^2}{4}\right] I$$

Thus

$$V^{2n} = \left[g^2\left(\hat{N} + \frac{1}{2}\right) + \frac{\Delta^2}{4}\right]^n I$$

$$V^{2n+1} = \left[g^2\left(\hat{N} + \frac{1}{2}\right) + \frac{\Delta^2}{4}\right]^n \begin{bmatrix} \frac{\Delta}{2} & ga \\ ga^{\dagger} & -\frac{\Delta}{2} \end{bmatrix}$$

finally, we obtain

$$U_I(t) = I\cos\Omega_N t - \mathrm{i}\frac{\sin\Omega_N t}{\Omega_N}\begin{bmatrix} \frac{\Delta}{2} & ga \\ ga^\dagger & -\frac{\Delta}{2} \end{bmatrix}, \quad \Omega_N = \sqrt{g^2\left(\hat{N}+\frac{1}{2}\right)+\frac{\Delta^2}{4}}$$

or in the matrix form

$$U_I(t) = \begin{bmatrix} \cos\widehat{\Omega}_{n+1}t - \mathrm{i}\frac{\Delta}{2\widehat{\Omega}_{n+1}}\sin\hat{\Omega}_{n+1}t & -\mathrm{i}g\frac{\sin\widehat{\Omega}_{n+1}t}{\Omega_{n+1}}a \\ -\mathrm{i}g\frac{\sin\widehat{\Omega}_n t}{\widehat{\Omega}_n}a^\dagger & \cos\widehat{\Omega}_n t + \mathrm{i}\frac{\Delta}{2\widehat{\Omega}_n}\sin\hat{\Omega}_n t \end{bmatrix} \tag{5.5}$$

where

$$\widehat{\Omega}_n = \sqrt{g^2\hat{n} + \Delta^2/4}, \quad \hat{n} = a^\dagger a \tag{5.6}$$

We would like to mention here the following convenient commutation relations:

$$f(\hat{n})a = af(\hat{n}-1), \quad f(\hat{n})a^\dagger = a^\dagger f(\hat{n}+1) \tag{5.7}$$

where $f(\hat{n})$ is an arbitrary function of the photon-number operator.

In the case of the exact resonance, Equation 5.5 simplifies to

$$U_I(t) = \begin{bmatrix} \cos gt\sqrt{\hat{n}+1} & -\mathrm{i}\frac{\sin gt\sqrt{\hat{n}+1}}{\sqrt{\hat{n}+1}}a \\ -\mathrm{i}\frac{\sin gt\sqrt{\hat{n}}}{\sqrt{\hat{n}}}a^\dagger & \cos gt\sqrt{\hat{n}} \end{bmatrix}$$

We can rewrite this formula in a different way using the phase operator for the field (see Section 3.7):

$$U_I(t) = \begin{bmatrix} \cos gt\sqrt{\hat{n}+1} & -\mathrm{i}\sin gt\sqrt{\hat{n}+1}\,\mathrm{e}^{\mathrm{i}\hat{\phi}} \\ -\mathrm{i}\sin gt\sqrt{\hat{n}}\,\mathrm{e}^{-\mathrm{i}\hat{\phi}} & \cos gt\sqrt{\hat{n}} \end{bmatrix} \tag{5.8}$$

Note that this form coincides with the atomic evolution operator in the external classical field (Section 3.1), if $\mathrm{e}^{\mathrm{i}\hat{\phi}}$ and $\hat{n} \approx \hat{n}+1$ are replaced by the classical field phase and amplitude.

When this evolution operator acts on the initial state, we obtain the time dependent wave function

$$|\psi(t)\rangle = U_I(t)\,|\psi(0)\rangle$$

For instance, for the initial state $|\psi(0)\rangle = |\phi\rangle_f |1\rangle_a$, where $|\phi\rangle_f = \sum_n \phi_n |n\rangle_f$ is an arbitrary field state, taking into account that

$$\cos gt\sqrt{\hat{n}+1}|\phi\rangle_f = \sum_n \phi_n \cos gt\sqrt{n+1}|n\rangle_f$$

$$\frac{\sin gt\sqrt{\hat{n}}}{\sqrt{\hat{n}}} a^\dagger |\phi\rangle_f = \sum_n \phi_n \sin gt\sqrt{n+1}|n+1\rangle_f$$

we obtain

$$U_I(t)\,|\phi\rangle_f|1\rangle_a = \sum_n \phi_n \cos gt\sqrt{n+1}|n\rangle_f|1\rangle_a - \mathrm{i}\sum_n \phi_{n-1}\sin gt\sqrt{n}|n\rangle_{|}0\rangle_a$$

The evolution operator enables us to find an arbitrary operator in the Heisenberg representation. For instance, for the atomic inversion operator σ_z under exact resonance, we have $\sigma_z(t) = U^\dagger \sigma_z U$, where

$$U^\dagger = \begin{bmatrix} \cos gt\sqrt{aa^\dagger} & \mathrm{i}a\dfrac{\sin gt\sqrt{a^\dagger a}}{\sqrt{a^\dagger a}} \\ \mathrm{i}a^\dagger \dfrac{\sin gt\sqrt{aa^\dagger}}{\sqrt{aa^\dagger}} & \cos gt\sqrt{a^\dagger a} \end{bmatrix}$$

Using the commutation relations (5.7) we arrive at

$$\sigma_z(t) = \begin{bmatrix} \cos 2gt\sqrt{\hat{n}+1} & -\mathrm{i}\dfrac{\sin 2gt\sqrt{\hat{n}+1}}{\sqrt{\hat{n}+1}}a \\ \mathrm{i}\dfrac{\sin 2gt\sqrt{\hat{n}}}{\sqrt{\hat{n}}}a^\dagger & -\cos 2gt\sqrt{\hat{n}} \end{bmatrix} \tag{5.9}$$

To find the atomic inversion evolution, one needs to calculate the average of the above operator for a given initial state. For instance, if we take the atom initially in its ground state, we obtain

$${}_a\langle 0|\sigma_z(t)|0\rangle_a = -\cos 2gt\sqrt{a^\dagger a}$$

which is still the field operator. For an arbitrary initial field state, we have $|\phi\rangle_f = \sum_{n=0}^{\infty}\phi_n|n\rangle_f$, and we arrive at the result

$${}_f\langle\phi|\,{}_a\langle 0|\sigma_z(t)|0\rangle_a\,|\phi\rangle_f = -\sum_n |\phi_n|^2 \cos 2gt\sqrt{n}$$

We conclude this section with the discussion of the case of a large detuning (which is also called the *dispersive limit*), $\Delta \gg g\sqrt{\overline{n}}$, where $\overline{n}$ is the average number of photons in the field. Expanding the operator $\widehat{\Omega}_n$ (5.6) in series with a small parameter $g\sqrt{n}/\Delta$

$$\widehat{\Omega}_n \approx \frac{\Delta}{2} + \frac{g^2 n}{\Delta} + \cdots \tag{5.10}$$

we obtain the following approximation for the evolution operator (5.5):

$$U_I(t) \approx \mathrm{e}^{-\mathrm{i}\xi t/2} \begin{bmatrix} \mathrm{e}^{-\mathrm{i}[\Delta/2+\chi(\hat{n}+1/2)]t} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}[\Delta/2+\chi(\hat{n}+1/2)]t} \end{bmatrix}$$

where $\chi = g^2/\Delta$. Since the evolution operator in the dispersive limit takes the diagonal form, the effect of the field on the atom reduces to phase shifts, different for the excited and ground atomic states. An arbitrary initial state

$$|\psi(0)\rangle = \begin{bmatrix} b \\ a \end{bmatrix} \sum_n \varphi_n |n\rangle_f, \quad |a|^2 + |b|^2 = 1$$

evolves as follows:

$$|\psi(t)\rangle = U_I(t)|\mathrm{in}\rangle_{\mathrm{at}}|\varphi\rangle_f = \mathrm{e}^{-\mathrm{i}\chi t/2} \begin{bmatrix} b\,\mathrm{e}^{-\mathrm{i}t(\Delta+\chi)/2} \sum_n \varphi_n\,\mathrm{e}^{-\mathrm{i}\chi nt}|n\rangle_f \\ a\,\mathrm{e}^{\mathrm{i}t(\Delta+\chi)/2} \sum_n \varphi_n\,\mathrm{e}^{\mathrm{i}\chi nt}|n\rangle_f \end{bmatrix}$$

In particular, in the case of a coherent initial state of the field, $|\varphi\rangle_f = |\alpha\rangle_f$, we obtain

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\chi t/2}\left[b\,\mathrm{e}^{-\mathrm{i}t(\Delta+\chi)/2}|\alpha\,\mathrm{e}^{-\mathrm{i}\chi nt}\rangle_f|1\rangle_{\mathrm{at}} + a\,\mathrm{e}^{\mathrm{i}t(\Delta+\chi)/2}|\alpha\,\mathrm{e}^{\mathrm{i}\chi nt}\rangle_f|0\rangle_{\mathrm{at}}\right]$$

The density matrix of the field takes on the form

$$\rho_f = \mathrm{Tr}_{\mathrm{at}}\left(|\psi(t)\rangle\langle\psi(t)|\right) = |b|^2|\alpha\,\mathrm{e}^{-\mathrm{i}\chi nt}\rangle_{ff}\langle\alpha\,\mathrm{e}^{-\mathrm{i}\chi nt}| + |a|^2|\alpha\,\mathrm{e}^{\mathrm{i}\chi nt}\rangle_{ff}\langle\alpha\,\mathrm{e}^{\mathrm{i}\chi nt}|$$

which represents a statistical mixture of two coherent states centered at $\alpha\,\mathrm{e}^{-\mathrm{i}\chi nt}$ and $\alpha\,\mathrm{e}^{\mathrm{i}\chi nt}$, respectively.

The effective interaction Hamiltonian for the JCM in the dispersive limit has the form

$$V \simeq \left[\frac{\Delta}{2} + \chi\left(\hat{n} + \frac{1}{2}\right)\right]\sigma_z + \frac{\chi}{2} \tag{5.11}$$

5.4
The Classical Field Limit

Let us introduce a new operator $\hat{Q}$ [46] writing in the bare atomic basis

$$\hat{Q} = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\hat{\phi}} & 0 \\ 0 & 1 \end{bmatrix} \tag{5.12}$$

where $\mathrm{e}^{\mathrm{i}\hat{\phi}}$ is the phase operator (3.126). It is well known that phase operators defined in this manner are not unitary, and, therefore, the operator $\hat{Q}$ is not unitary:

$$\left[\hat{Q}, \hat{Q}^{\dagger}\right] = |0\rangle_{ff}\langle 0|\,\sigma_+\sigma_-$$

However, the operator $\hat{Q}$ is unitary on the field states $|n\rangle_f$ if $n \geq 1$, i.e. if one can neglect the contribution of the vacuum state. Another representation of $\hat{Q}$ is

$$\hat{Q} = \mathrm{e}^{(1/2)\mathrm{i}\hat{\phi}(\sigma_z+1)} = \sum_{k=0,1} \mathrm{e}^{\mathrm{i}\hat{\phi}k} |k\rangle_{a\,a}\langle k| \tag{5.13}$$

Assuming that the Fock states with small photon numbers do not contribute significantly to the initial field state, we apply the $\hat{Q}$ transformation to the JCM Hamiltonian in the exact resonance case

$$\hat{Q}^\dagger \hat{n} \hat{Q} = \hat{n} - \frac{\sigma_z + 1}{2}, \quad \hat{Q}^\dagger \sigma_z \hat{Q} = \sigma_z$$

and

$$\widetilde{H}_0 = \hat{Q}^\dagger H_0 \hat{Q} = \omega \hat{n}$$

On the other hand,

$$\hat{Q}^\dagger \sigma_\pm \hat{Q} = \sigma_\pm \, \mathrm{e}^{\mp \hat{\phi}}$$

and the transformed interaction Hamiltonian takes the following form:

$$\widetilde{V} = \hat{Q}^\dagger V \hat{Q} = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\hat{\phi}} & 0 \\ 0 & 1 \end{bmatrix} g \begin{bmatrix} 0 & a \\ a^\dagger & 0 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\mathrm{i}\hat{\phi}} & 0 \\ 0 & 1 \end{bmatrix} = g \begin{bmatrix} 0 & \mathrm{e}^{-\mathrm{i}\hat{\phi}} a \\ a^\dagger \, \mathrm{e}^{\mathrm{i}\hat{\phi}} & 0 \end{bmatrix}$$

Finally, due to $\mathrm{e}^{-\mathrm{i}\hat{\phi}} a = a^\dagger \, \mathrm{e}^{\mathrm{i}\hat{\phi}} = \sqrt{a^\dagger a}$, we obtain

$$\widetilde{V} = g \begin{bmatrix} 0 & \sqrt{a^\dagger a} \\ \sqrt{a^\dagger a} & 0 \end{bmatrix} = g \sqrt{a^\dagger a} \, \sigma_x$$

The transformed Hamiltonian is a diagonal operator in the field space. The $\hat{n}$ operator commutes with all the other ingredients of the transformed Hamiltonian and may be treated as the c-number when calculating the transformed evolution operator $\tilde{U}(t) = \exp\left(-\mathrm{i}\widetilde{H}t\right)$:

$$\widetilde{H} = \widetilde{H}_0 + \widetilde{V} = \omega \hat{n} + 2g\sqrt{\hat{n}}\,\sigma_x$$
$$\tilde{U}(t) = \exp(-\mathrm{i}\omega\hat{n}) \exp(-2\mathrm{i}gt\sqrt{\hat{n}}\,\sigma_x)$$

or, in the matrix form

$$\tilde{U}(t) = \exp\left(-\mathrm{i}\omega\hat{n}\right) \begin{bmatrix} \cos gt\sqrt{\hat{n}} & -\mathrm{i}\sin gt\sqrt{\hat{n}} \\ -\mathrm{i}\sin gt\sqrt{\hat{n}} & \cos gt\sqrt{\hat{n}} \end{bmatrix}$$

This coincides with the atomic evolution operator in the classical field of intensity $\hat{n}$ and phase equal to zero. Finally, one can recover the quantum field evolution operator (5.8) by making the inverse transformation,

$$U(t) = \hat{Q}\tilde{U}(t)\hat{Q}^\dagger \tag{5.14}$$

We have thus reconstructed the exact evolution operator for a two-level atom interacting with a quantum field mode, knowing the functional form of the atomic evolution operator in an external classical field. Surprisingly, this procedure can be generalized to the case of an arbitrary atomic system interacting with a quantum cavity field, provided that the field energy is much larger then the energy of the atomic system. In this general case, the quantum field evolution operator can be approximately reconstructed knowing the functional form of the corresponding atomic evolution operator in an external classical field [46]. One may speak, in this sense, about the ''semiclassical quantization'' of the atomic evolution operator.

5.5
Collapses and Revivals

Collapses and revivals of oscillations of the atomic inversion $\langle\sigma_z(t)\rangle$ represent one of the most important features of the JCM which, in fact, motivated considerable attention to this model in the 1980s [47, 47a]. Consider the atom initially in one of its bare states ($|k\rangle_a$, $k = 1$ or 0) and the field in a coherent state (CS) with a high average photon number $\overline{n} \gg 1$ (for instance, $\overline{n} = 100$ may be considered as a very ''high'' photon number in this case). Then the field phase is well defined: $\overline{\Delta\phi} \sim 1/2\sqrt{\overline{n}}$. Fluctuations of the photon number grow with $\overline{n}$, $\overline{\Delta n} = \sqrt{\langle n^2\rangle - \overline{n}^2} \sim \sqrt{\overline{n}}$; however, the relative fluctuations diminish, $\overline{\Delta n}/\overline{n} \sim 1/\sqrt{\overline{n}}$. Thus, one can say that the amplitude of the field is relatively well defined. Therefore, one can expect that the interaction with a strong coherent field leads to oscillations of the atomic inversion with the frequency $2\overline{\Omega} = 2g\sqrt{\overline{n}}$, as it happens in the case of the external classical field (Rabi oscillations). However, the behavior of the atomic inversion is much more complicated. $\langle\sigma_z(t)\rangle$ oscillates with the Rabi frequency only for very short times; at times $gt \sim 1$ oscillations collapse and the atomic inversion remains constant for a large period of time; however the oscillations revive at the time $gT_{\mathrm{r}} \sim 2\pi\sqrt{\overline{n}}$. This process is repeated several times with successive revivals becoming less and less pronounced; finally, they are changed into a ''quasi-chaotic behavior''. This simple model reveals clearly the difference between classical and quantum field dynamics.

We now present a qualitative discussion of this phenomenon [47]. Let us consider the atom initially in its ground state and the field in a coherent state $|\alpha\rangle_f$ with $\overline{n} = |\alpha|^2 \gg 1$. The atomic inversion is determined by the formula

$$\langle\sigma_z\,(t)\rangle = \langle\psi(t)|\sigma_z(0)|\psi(t)\rangle$$

where

$$|\psi(t)\rangle = U(t)\,|\alpha\rangle_f \otimes |0\rangle_a$$

From Equation 5.5, we obtain

$$\langle\sigma_z\,(t)\rangle = -\sum_{n=0}^{\infty} P_n\left[\frac{\Delta^2}{4\Omega_n^2} + \left(1 - \frac{\Delta^2}{4\Omega_n^2}\right)\cos 2\Omega_n t\right] \tag{5.15}$$

where $P_n = \mathrm{e}^{-\overline{n}}\overline{n}^n/n!$ is the Poisson distribution and

$$\Omega_n = \sqrt{\frac{\Delta^2}{4} + g^2 n}$$

The frequencies Ω_n are often called *the quantum Rabi frequencies*. Recall that the Poisson distribution has a relatively sharp peak in the vicinity of the point $n \sim \overline{n}$ with the spread $\overline{\Delta n} \sim \sqrt{\overline{n}}$. Therefore, the frequencies that essentially contribute to the sum (5.15) correspond to values of n from the interval $\overline{n} - \overline{\Delta n} < n < \overline{n} - \overline{\Delta n}$, where we may use the expansion of quantum Rabi frequencies around the value of the classical Rabi frequency $\overline{\Omega} = \Omega\,(\overline{n}) = \sqrt{\Delta^2/4 + g^2\overline{n}}$:

$$\Omega_n = \sqrt{\frac{\Delta^2}{4} + g^2 n} = \overline{\Omega} + \frac{g^2 \Delta n}{2\overline{\Omega}} - \frac{(g^2\Delta n)^2}{8\overline{\Omega}^3} + \cdots \tag{5.16}$$

Hence, the spread of the frequencies is of the order

$$\Delta\Omega_n \sim \frac{g^2\overline{\Delta n}}{\overline{\Omega}} \sim \frac{g}{\sqrt{1 + \Delta^2/4g^2\overline{n}}}$$

This means that the cosine functions in the sum (5.15) become out of phase at time $T_\mathrm{c} \sim 1/\Delta\Omega$ and oscillations collapse. The collapse time is given by

$$T_\mathrm{c} \sim \frac{1}{\Delta\Omega} = \frac{\sqrt{1 + \Delta^2/(4g^2\overline{n})}}{g}$$

On the other hand, oscillations with neighboring frequencies become in phase again at time instants when

$$2\pi k = 2t\,[\Omega\,(\overline{n}+1) - \Omega\,(\overline{n})] \approx \frac{g^2 t}{\overline{\Omega}}, \quad k = 1, 2, \ldots$$

i.e. the system revives at the following instants of time:

$$T_\mathrm{r} \sim k\,\frac{2\pi\overline{\Omega}}{g^2}$$

Therefore, there are three different time scales in the JCM: the period of a single Rabi oscillation $T_{\mathrm{Rabi}} \sim 1/2\overline{\Omega}$ (which is very short if $\overline{n} \gg 1$); the collapse time $gT_\mathrm{c} \sim \sqrt{1 + \Delta^2/(4g^2\overline{n})}$; and the revival time $T_\mathrm{r} \sim 2\pi\overline{\Omega}/g^2$. It is clear that $T_{\mathrm{Rabi}} \ll T_\mathrm{c} \ll T_\mathrm{r}$. These timescales are well seen in Figure 5.1 where the typical time evolution of the atomic inversion, $\langle\sigma_z\,(t)\rangle$, is shown.

Let us turn to the quantitative description of the problem considering separately two different limits–of a large detuning $\Delta \gg g\sqrt{\overline{n}}$ (the dispersive limit) and of zero detuning $\Delta = 0$.

5.5.1
The Dispersive Limit

In the case of large detuning, $\Delta \gg g\sqrt{\overline{n}}$, one can expand the quantum Rabi frequencies, Ω_n, in a power series with a small parameter $g\sqrt{\overline{n}}/\Delta$ (in the same manner as was done in Equation 5.10). In this case, we restrict ourselves to the

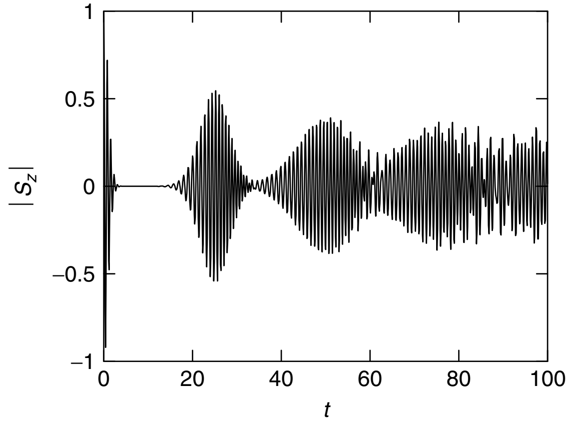

Figure 5.1 Collapses and revivals of the atomic inversion, $\langle\sigma_z(t)\rangle$in the JCM. The atom is initially excited and the field is taken in the initial coherent state with $\overline{n} = 16$.

linear approximation for the frequencies:

$$\Omega_n \approx \frac{\Delta}{2} + \xi n + \cdots, \quad \xi = \frac{g^2}{\Delta}$$

Then the sum in Equation 5.15 is transformed to the form

$$\left\langle \sigma_z(t) \right\rangle = -1 + \frac{4\xi}{\Delta}\left[\overline{n} - \sum_{n=0}^{\infty} P_n n \cos\left(\Delta + 2\xi n\right) t\right]$$

where the terms of order $O\left((\xi\overline{n}/\Delta)^2\right)$ have been neglected. This sum contains harmonic frequencies and can be easily calculated. One finds

$$\begin{aligned}\sum_n P_n n\, \mathrm{e}^{\mathrm{i}2\xi n t} &= \overline{n}\,\mathrm{e}^{-\overline{n}}\frac{\partial}{\partial \overline{n}}\sum_n \frac{\left[\overline{n}\exp\left(\mathrm{i}2\xi t\right)\right]^n}{n!} \\ &= \overline{n}\,\mathrm{e}^{\overline{n}(\cos 2\xi t - 1)}\,\mathrm{e}^{\mathrm{i}(\overline{n}\sin 2\xi t + 2\xi t)}\end{aligned}$$

Here, the envelope of the Rabi oscillations $\exp\left(-2\overline{n}\sin^2\xi t\right)$ diminishes when $2\overline{n}\sin^2\xi t \sim 2\overline{n}(\xi t)^2 \sim 1$, which determines the collapse time

$$T_{\mathrm{c}} \sim \frac{1}{\sqrt{2\overline{n}}\xi} = \frac{\Delta}{g^2\sqrt{2\overline{n}}}$$

The function $\langle\sigma_z(t)\rangle$ does not oscillate when $\overline{n}\sin^2(\xi t) \gg 1$, and then oscillations revive again at the time instants $\xi t \sim \pi k,\ k = 1, 2, \ldots$, which gives the revival times

$$kT_{\mathrm{r}} = \frac{k\pi\Delta}{g^2}$$

The phase of the oscillations $\approx \Delta + \overline{n}\sin 2\xi t$, in the vicinities of the revival times $kT_r - T_c < t < kT_r + T_c$ can be approximated as $2\overline{\Omega}t - 2\pi k\overline{n}$, which leads us to the expression

$$\langle \sigma_z(t) \rangle \approx -1 + \frac{4\xi\overline{n}}{\Delta}\left[1 - e^{-2\overline{n}\sin^2 \xi t}\cos\left(2\overline{\Omega}t - 2\pi k\overline{n}\right)\right]$$

$$\overline{\Omega} = \frac{\Delta}{2} + \xi\overline{n}$$

In this case, the behavior of the atomic inversion is regular. Rabi oscillations (of a very small amplitude) revive periodically and subsequent revivals are identical to each other. It is a general property of the harmonic approximation that works well in the case under study. Note, that the revival times do not depend on the field intensity in this limit.

5.5.2
Exact Resonance

Replacing $\Delta = 0$ in Equation 5.15 we obtain

$$\langle \sigma_z(t) \rangle = -\sum_{n=0}^{\infty} P_n \cos 2gt\sqrt{n} \tag{5.17}$$

This sum is essentially anharmonic, and it is necessary to take into account the quadratic terms in the expansion for the Rabi frequencies (Equation 5.16). Different analytic methods were applied to find a closed expression for this sum in the strong field case, ($\overline{n} \gg 1$), in Refs. [47, 48]. All these methods lead to similar (though different in small details) expressions for the atomic inversion as a sum of oscillating packets with Gaussian envelopes. Each packet corresponds to a single revival of the Rabi oscillations with growing spreads and decreasing maxima of subsequent revivals. Here we show that the sum (5.17) can be approximately written in terms of the Jacobi function Θ_3 (see, e.g. [49]). Once this is done, the analytic expression for the collapses and revivals arises naturally, by using the well-known properties of the Θ_3 function. In fact, this is a general method of treating sums with quadratic frequencies.

The sum (5.17) can be rewritten in the form

$$\langle \sigma_z(t) \rangle = -\text{Re}\, W_+, \quad W_+ = \sum_{n=0} P_n\, e^{i2gt\sqrt{n}}$$

Let us start with the case of the integer average photon number $\overline{n}$. Expanding the square root in a Taylor series and taking into account the second-order terms, we get

$$\sqrt{n} \approx \sqrt{\overline{n}} + \frac{m}{2\sqrt{\overline{n}}} - \frac{m^2}{8\overline{n}^{3/2}}, \quad m = n - \overline{n}$$

where m is integer. For strong fields ($\overline{n} \gg 1$), one may, with a high accuracy, approximate the Poisson distribution by a Gaussian distribution

$$P_n \approx \frac{1}{\sqrt{2\pi\overline{n}}}\,\mathrm{e}^{-m^2/2\overline{n}}$$

Then the sum W_+ takes the form of the Jacobi Θ_3 function

$$W_+ = \frac{\mathrm{e}^{\mathrm{i}gt2\sqrt{\overline{n}}}}{\sqrt{2\pi\overline{n}}}\,\Theta_3(z|\mu)$$

where, by definition,

$$\Theta_3(z|\mu) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}zm-\mu m^2}$$

and

$$z = \frac{gt}{\sqrt{\overline{n}}}, \quad \mu = \frac{1}{2\overline{n}}\left(1 + \frac{\mathrm{i}gt}{2\sqrt{\overline{n}}}\right), \quad \sigma = 2\overline{n}\mu = 1 + \frac{\mathrm{i}gt}{2\sqrt{\overline{n}}}$$

Using the Poisson summation formula,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \int_{\infty}^{\infty} \mathrm{d}x\, f(x)\,\mathrm{e}^{2\pi\mathrm{i}mx}$$

we obtain another representation for the Jacobi function

$$\Theta_3(z|\mu) = \sqrt{\frac{\pi}{\mu}} \sum_k \exp\left[-\frac{(z-2\pi m)^2}{4\mu}\right]$$

The sum W_+ can thus be approximated as

$$W_+ = \frac{\mathrm{e}^{\mathrm{i}gt2\sqrt{\overline{n}}}}{\sqrt{\sigma}} \sum_k \exp\left[-\frac{g^2(t-T_\mathrm{r}k)^2}{2\sigma}\right], \quad T_\mathrm{r} = 2\pi\sqrt{\overline{n}}/g$$

Taking into account that

$$\frac{1}{\sigma} = \frac{\mathrm{e}^{-\mathrm{i}\phi}}{|\sigma|}, \quad \phi = \arctan\frac{gt}{2\sqrt{\overline{n}}}, \quad |\sigma|^2 = \left[1 + \frac{g^2t^2}{4\overline{n}}\right]$$

we obtain

$$\langle\sigma_z(t)\rangle = -\mathrm{Re}\, W_+ = |\sigma|^{-1/2} \sum_k \mathrm{e}^{-\tau_k^2/2|\sigma|^2} \cos\left[2g\sqrt{\overline{n}}t + \frac{\tau_k^2 gt}{4\sqrt{\overline{n}}|\sigma|^2} - \frac{\phi}{2}\right]$$

Here we have introduced the local time in the vicinity of the kth revival, $\tau_k = g(t - T_\mathrm{r}k)$. The spread of each revival, $\overline{\Delta\tau_k}/g$ has the same order as the collapse time T_c. Recalling that the oscillations disappear between different revivals, we can use the values of the parameters $\sigma(t)$ and $\phi(t)$ at revival times $t = T_\mathrm{r}k$ (indeed, these are slowly varying functions, which do not change significantly in the time range

$kT_r - T_c < \tau_k < kT_r + T_c)$

$$|\sigma(t = T_r k)|^2 \to 1 + \pi^2 k^2, \quad \phi(t = T_r k) \to \phi_k = \arctan \pi k$$

Finally, we obtain

$$\langle \sigma_z (t) \rangle = -\sum_k \exp\left\{ -\frac{\tau_k^2}{2(1+\pi^2 k^2)} \right\} \frac{\cos\left[\psi_k(t)\right]}{(1+\pi^2 k^2)^{1/4}} \tag{5.18}$$

$$\psi_k(t) = 2g\sqrt{\overline{n}}t + \frac{\tau_k^2 \pi k}{2(1+\pi^2 k^2)} - \frac{\phi_k}{2} \tag{5.19}$$

In this sum, the index k corresponds to the revival number, as the kth term is essentially different from zero only in the vicinity of the revival time $T_r k$. For instance, within the interval of the initial collapse, $\tau_0 = gt \sim 1 \ll 2\pi\sqrt{\overline{n}}$ only the term $k = 0$ is different from zero in the sum (5.18). In turn, the first revival appears when $t \sim T_r$ and $\tau_1 \sim 1 \ll 2\pi\sqrt{\overline{n}}$. Revivals have Gaussian envelopes with a spread $|\sigma(kT_r)|^2 = 1 + \pi^2 k^2$ and a decreasing maximum $(1 + \pi^2 k^2)^{-1/4}$, which means that every subsequent revival is wider and lower than the preceding one. The Rabi oscillations of the frequency $2g\sqrt{\overline{n}}$ inside every revival are modulated by the phase shift $\tau_k^2 \pi k/2(1 + \pi^2 k^2) - \phi_k/2$.

For noninteger values of the average photon number $\overline{n}$, the above formulas must be modified in such a way that the values of the index m in the definition of the Jacobi function are integers. To achieve this, we introduce

$$\overline{n} = [\overline{n}] + \delta, \quad n = [\overline{n}] + m = \overline{n} - \delta + m$$

where $[\overline{n}]$ is the integer part of $\overline{n}$. Thus we have

$$P_m \approx \frac{1}{2\pi\overline{n}} \exp\left(-\frac{(m-\delta)^2}{2\overline{n}} \right)$$

and

$$\sqrt{n} = \sqrt{[\overline{n}]} + \frac{m}{2\sqrt{\overline{n}}} - \frac{m^2}{8\overline{n}^{3/2}} + O(1/\overline{n})$$

Correspondingly, the phase $\psi_k(t)$ in Equation 5.19 must be replaced by the phase

$$\psi_k(t) = 2g\sqrt{[\overline{n}]}t + \frac{\tau_k^2 \pi k}{2(1+\pi^2 k^2)} - \frac{\phi_k}{2}$$

5.6
The JCM with an Initial Thermal Field

Let us consider now the JCM with the initial field in a thermal state. If the atom is taken initially in its ground state, the atomic inversion is given by the same

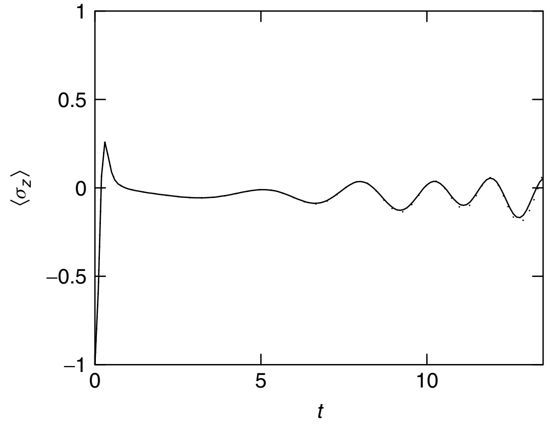

Figure 5.2 Dynamics of the atomic inversion in the JCM with the initial thermal field state, $\overline{n} = 25$, and the atom initially in its ground state. The solid line corresponds to the numerical calculation and the dotted line to the analytic approximation.

Equation 5.17, and the photon-number distribution for the thermal state has a form

$$P_n = (1 - q)\, q^n, \quad q = \frac{\overline{n}}{\overline{n} + 1} \tag{5.20}$$

and $\overline{n}$ is the average photon number. In this case, the collapse–revival structure disappears [50] (see, also, [51]). The initial atomic inversion collapses to the midlevel and does not oscillate for some period of time; this is followed by a quasi-chaotic behavior (see Figure 5.2).

To evaluate the series (Equation 5.17) we use the Abel–Plana summation formula (see, [52], Chapter VII, Miscellaneous examples, example 7):

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \int_0^{\infty} \mathrm{d}x f(x) + \mathrm{i} \int_0^{\infty} \mathrm{d}x \frac{f(\mathrm{i}x) - f(-\mathrm{i}x)}{\exp(2\pi x) - 1}$$

For atomic inversion, we find

$$\langle \sigma_z(t) \rangle = -(1 - q) \left(\frac{1}{2} + I_0(t) + I_1(t) \right) \tag{5.21}$$

where

$$\begin{aligned} I_0(t) &= \mathrm{Re} \int_0^{\infty} \mathrm{d}x \, \exp\left[x \ln q + 2\mathrm{i}gt\sqrt{x} \right] = \\ &= \frac{1}{|\ln q|} \left[1 + \sqrt{\pi} z \, \mathrm{e}^{-z^2} \mathrm{Im}\left(1 - \mathrm{erf}(-\mathrm{i}z)\right) \right] \\ I_1(t) &= \int_0^{\infty} \mathrm{d}x \, \frac{\left[\chi(t, x) + \chi(-t, x) \right]}{\exp(2\pi x) - 1}, \quad z = \frac{2gt}{\sqrt{|\ln q|}} \end{aligned} \tag{5.22}$$

and

$$\chi(t, x) = \exp\left(gt\sqrt{2x}\right) \sin\left(gt\sqrt{2x} - x \ln q\right)$$

The first integral I_0 in Equation 5.21 corresponds to the semiclassical limit when the series is replaced by the integral and describes the initial part of the curve (initial collapse) [50]. In the strong field case, $\overline{n} \gg 1$ and $q = 1 - \epsilon$, $-\ln q \approx \epsilon \sim 1/\overline{n}$. Then, for short times, $gt \sim \sqrt{\epsilon}$, the integral $I_0 \sim 1/\epsilon$, while for $gt > \sqrt{\epsilon}$ it rapidly tends to zero ($\sim (gt)^{-2}$). Note that the average photon number $\overline{n}$ appears only in the scale factors in Equation 5.22, which determines the scales of time and of I_0 itself.

The second integral I_1 gives quantum corrections that are responsible for quasi-chaotic behavior. The principal contribution to the integral is given by the vicinity of the point $x_0 = (gt)^2/(8\pi^2)$ (of the width $\Delta x \sim gt\pi^{-3/2}/2$). If $x_0\epsilon \ll 1$, which means that $gt \ll \pi\sqrt{8\overline{n}}$, the integral I_1 is almost independent of the average photon number and is approximated as follows:

$$I_1(t) = 2\int_0^\infty \mathrm{d}x \, \frac{\sinh\left(gt\sqrt{2x}\right) \sin\left(gt\sqrt{2x}\right)}{\exp(2\pi x) - 1} + O(\epsilon) \tag{5.23}$$

In Figure 5.2 we show the results of both the exact (numerical) and analytical calculations, Equations 5.21–5.23.

To conclude, the quantum corrections to the semiclassical dynamics of the atomic inversion in the strong thermal field do not depend on the intensity of the field in the large time range $gt \ll \pi\sqrt{8\overline{n}}$ [53]. This range corresponds to the beginning of the "quasi-chaotic" behavior.

5.7
Trapping States

As has been seen, the behavior of the atom interacting with a quantum field is different from that in a classical field. However, there exist special initial states of the atom for which the quantum nature of the field almost does not affect the atomic dynamics [54].

In the classical field limit, the photon creation and annihilation operators in the JCM Hamiltonian should be replaced by complex numbers (field amplitudes)

$$\hat{a} \to \alpha \equiv \sqrt{\overline{n}}\,\mathrm{e}^{\mathrm{i}\varphi}, \quad \hat{a}^\dagger \to \overline{\alpha} \equiv \sqrt{\overline{n}}\,\mathrm{e}^{-\mathrm{i}\varphi} \tag{5.24}$$

where $\overline{n}$ is the classical field intensity in units of the photon number and φ is the classical field phase. The resulting Hamiltonian describes the atom in an external classical field of amplitude $\sqrt{\overline{n}}$ and phase φ:

$$H_{\mathrm{cl}} = g\sqrt{\overline{n}}\left(\mathrm{e}^{-\mathrm{i}\varphi}\sigma_- + \mathrm{e}^{\mathrm{i}\varphi}\sigma_+\right) = g\begin{bmatrix} 0 & \alpha \\ \overline{\alpha} & 0 \end{bmatrix} \tag{5.25}$$

The atomic evolution operator in the external classical field takes the form

$$U_{\mathrm{cl}}(t) = \begin{bmatrix} \cos(\Omega t) & -\mathrm{i}\sin(\Omega t)\,\mathrm{e}^{\mathrm{i}\varphi} \\ -\mathrm{i}\sin(\Omega t)\,\mathrm{e}^{-\mathrm{i}\varphi} & \cos(\Omega t) \end{bmatrix}$$

where $\Omega = g\sqrt{\bar{n}}$ is the classical Rabi frequency. Let us denote the eigenvalues and eigenstates of H_{cl} as $\lambda_p\sqrt{\bar{n}}$ and $|\underline{p,\varphi}\rangle_a$:

$$H_{\text{cl}}|\underline{p,\varphi}\rangle_a = \lambda_p\sqrt{\bar{n}}|\underline{p,\varphi}\rangle_a, \quad p = 0, 1 \tag{5.26}$$

where the argument φ indicates the dependence of the eigenvectors of the semiclassical Hamiltonian of the phase of this classical field. H_{cl} can be rewritten in the form

$$H_{\text{cl}} = \sqrt{\bar{n}}\,\mathrm{e}^{\mathrm{i}(1/2)\varphi\sigma_z}\,(\sigma_- + \sigma_+)\,\mathrm{e}^{-\mathrm{i}(1/2)\varphi\sigma_z}$$

and hence

$$|\underline{p,\varphi}\rangle_a = \mathrm{e}^{\mathrm{i}(1/2)\varphi\sigma_z}|\underline{p}\rangle_a \tag{5.27}$$

where $|\underline{p}\rangle_a = |\underline{p,0}\rangle_a$ are eigenvectors of the operator $\sigma_x = \sigma_- + \sigma_+$

$$\sigma_x|\underline{p}\rangle_a = \lambda_p|\underline{p}\rangle_a, \quad \lambda_p = 2p - 1, \quad p = 0, 1 \tag{5.28}$$

In the standard representation, the normalized eigenvectors $|\underline{p}\rangle_a$ are

$$|\underline{0}\rangle_a = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad |\underline{1}\rangle_a = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the vectors $|\underline{p,\varphi}\rangle_a$ have the form

$$|\underline{0,\varphi}\rangle_a = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathrm{e}^{\mathrm{i}\varphi/2} \\ -\mathrm{e}^{-\mathrm{i}\varphi/2} \end{bmatrix}, \quad |\underline{1,\varphi}\rangle_a = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathrm{e}^{\mathrm{i}\varphi/2} \\ \mathrm{e}^{-\mathrm{i}\varphi/2} \end{bmatrix}$$

We call the vectors $|\underline{p,\varphi}\rangle_a$ semiclassical states. Their evolution under the classical field Hamiltonian is reduced to the global phase factor

$$\exp\left(-\mathrm{i}tH_{\text{cl}}\right)|\underline{p,\varphi}\rangle_a = \mathrm{e}^{-\mathrm{i}\lambda_p\sqrt{\bar{n}}t}|\underline{p,\varphi}\rangle_a$$

This means that atomic inversion will be always zero in these states, since the operator σ_z has an off-diagonal form in the basis of eigenvectors of the operator σ_x: $\langle\underline{p}|\sigma_z|\underline{p}\rangle = 0$.

Let us return now to the quantum field case and calculate the average value of the atomic inversion

$$\langle\sigma_z(t)\rangle = \langle\psi(t)|\sigma_z(0)|\psi(t)\rangle$$

when the initial field is in a coherent state $|\alpha\rangle_f$, $\alpha = \sqrt{\bar{n}}\,\mathrm{e}^{\mathrm{i}\phi}$ with high photon number, $\bar{n} \gg 1$ and the atom is initially taken to be in its semiclassical state, $|\underline{p,\varphi}\rangle_a$, for instance $|\underline{1,\varphi}\rangle_a$:

$$|\text{in}\rangle = |\alpha\rangle_f \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} \mathrm{e}^{\mathrm{i}\varphi/2} \\ \mathrm{e}^{-\mathrm{i}\varphi/2} \end{bmatrix} \tag{5.29}$$

The evolution of this state for the exact resonance case can be found applying the evolution operator (5.8) to the state (5.29):

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos gt\sqrt{\hat{n}+1}\,\mathrm{e}^{\mathrm{i}(1/2)\varphi} - \mathrm{i}\sin gt\sqrt{\hat{n}+1}\,\mathrm{e}^{\mathrm{i}\left(\hat{\phi}-(1/2)\varphi\right)} \\ \cos gt\sqrt{\hat{n}}\,\mathrm{e}^{-\mathrm{i}(1/2)\varphi} - \mathrm{i}\sin gt\sqrt{\hat{n}}\,\mathrm{e}^{-\mathrm{i}\left(\hat{\phi}-(1/2)\varphi\right)} \end{bmatrix}|\alpha\rangle$$

Recall that $\mathrm{e}^{\pm\mathrm{i}\hat{\phi}}$ are the field phase operators. In the coherent state, the phase fluctuations are $\sim 1/2|\alpha|$ and the phase is well defined when $\overline{n} = |\alpha|^2 \gg 1$. In other words,

$$\mathrm{e}^{\pm\mathrm{i}(1/2)\hat{\phi}}|\alpha\rangle = \mathrm{e}^{\pm\mathrm{i}(1/2)\varphi}|\alpha\rangle + O\left(\frac{1}{\sqrt{\overline{n}}}\right)$$

Hence we have

$$|\psi(t)\rangle \approx \frac{1}{\sqrt{2}}\begin{bmatrix} \mathrm{e}^{-\mathrm{i}gt\sqrt{\hat{n}+1}}\,\mathrm{e}^{\mathrm{i}(1/2)\varphi} \\ \mathrm{e}^{-\mathrm{i}gt\sqrt{\hat{n}}}\,\mathrm{e}^{-\mathrm{i}(1/2)\varphi} \end{bmatrix}|\alpha\rangle + O(\overline{n}^{-1/2}) \tag{5.30}$$

and the atomic inversion is approximately equal to zero.

$$\langle\sigma_z(t)\rangle = \langle\psi(t)|\sigma_z(0)\,|\psi(t)\rangle = O\left(\overline{n}^{-1/2}\right) \to 0, \quad \overline{n} \to \infty$$

This means that the states $|\underline{p,\varphi}\rangle_a$ are "trapping states"; there is no energy exchange between the atom and the field within an accuracy of $O\left(1/\sqrt{\overline{n}}\right)$. In this sense, the initial states (5.29) evolve as in the case of an external classical field. That is why the states

$$|\underline{p,\varphi}\rangle_a|\alpha\rangle_f = \mathrm{e}^{\mathrm{i}1/2\varphi\sigma_z}|\underline{p}\rangle_a|\alpha\rangle_f \tag{5.31}$$

are called *semiclassical states of the whole system.*

5.8
Factorization of the Wave Function

The semiclassical states are, in fact, more than trapping states. To see this, let us further transform the wave function (5.30). Expanding the square root $\sqrt{\hat{n}+1}$ in a power series with a small parameter $1/\overline{n} \ll 1$:

$$\sqrt{\hat{n}+1} \approx \sqrt{\hat{n}} + \frac{1}{2\sqrt{\hat{n}}} + O(\overline{n}^{-3/2})$$

we rewrite Equation 5.30 in the form

$$|\psi(t)\rangle \approx \frac{1}{\sqrt{2}}\,\mathrm{e}^{-\mathrm{i}gt\sqrt{\hat{n}}}\begin{bmatrix} \mathrm{e}^{-\mathrm{i}gt/(2\sqrt{\overline{n}})}\,\mathrm{e}^{\mathrm{i}(1/2)\varphi} \\ \mathrm{e}^{-\mathrm{i}(1/2)\varphi} \end{bmatrix}|\alpha\rangle$$

where we have substituted $1/\sqrt{\hat{n}} \to 1/\sqrt{\overline{n}}$. This leads to an error in the frequencies of the order $O(1/\overline{n})$. Indeed,

$$\frac{1}{\sqrt{\hat{n}}} = \frac{1}{\sqrt{\overline{n}+\Delta\hat{n}}} \approx \frac{1}{\sqrt{\overline{n}}} - \frac{\Delta\hat{n}}{2\overline{n}^{3/2}}, \quad \Delta\hat{n} = \hat{n} - \overline{n} \sim \sqrt{\overline{n}}$$

Therefore, our approximation is valid for times $gt < \overline{n}$. Now, taking into account that

$$\exp\left[-\mathrm{i}\frac{\omega_1 t}{2}\left(\sigma_z+1\right)\right]|\underline{1}\rangle_a = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathrm{e}^{-\mathrm{i}\omega_1 t} \\ 1 \end{bmatrix}, \qquad \omega_1 = \frac{g}{2\sqrt{\overline{n}}}$$

we conclude that the wave function of the system for the initial state (5.29) is approximately written in the following form [55]:

$$|\psi(t)\rangle \approx |A(t)\rangle_a |\Phi(t)\rangle_f = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathrm{e}^{-\mathrm{i}\omega_1 t} \\ 1 \end{bmatrix} \otimes \mathrm{e}^{-\mathrm{i}gt\sqrt{\hat{n}}}|\alpha\rangle$$

In general, for the initial atomic state $|\underline{p,\varphi}\rangle (p = 0, 1)$, the result reads

$$|\psi(t)\rangle \approx |A_p(t)\rangle_a |\Phi_p(t)\rangle_f \tag{5.32}$$

$$|A_p(t)\rangle_a = \exp\left[-\mathrm{i}\omega_p t\left(\sigma_z+1\right)/2\right]|\underline{p,\varphi}\rangle_a \tag{5.33}$$

$$|\Phi(t)\rangle_f = \mathrm{e}^{-\mathrm{i}\lambda_p gt\sqrt{\hat{n}}}|\alpha\rangle \tag{5.34}$$

where

$$\omega_p = \frac{\lambda_p g}{2\sqrt{\overline{n}}} \tag{5.35}$$

and λ_p is an eigenvalue of the semiclassical Hamiltonian (5.26) and (5.28).

Equation 5.32 implies that if the system is initially prepared in its semiclassical state (5.31), then the total wave function remains approximately factorized into the product of the atomic and field wave functions [55]. Usually, the factorization means that the subsystems do not interact. This is not our case, however, since the evolution of the atomic part of the wave function depends on the initial field intensity $\overline{n}$ and the field part depends on the initial atomic parameter λ_p. Moreover, the evolution of every subsystem (field or atom) alone is not unitary: the states $|\underline{1,\varphi}\rangle_a$, $|\underline{0,\varphi}\rangle_a$ evolve in such a manner that they coincide (up to the phase factor) at a specific instant of time $t_0 = \pi/\omega_1 = \pi\sqrt{\overline{n}}/g = T_\mathrm{r}/2$.

$$|A_1(t_0)\rangle_a = \frac{1}{\sqrt{2}}\begin{bmatrix} -\mathrm{i}\,\mathrm{e}^{\mathrm{i}(1/2)\varphi} \\ \mathrm{e}^{-\mathrm{i}(1/2)\varphi} \end{bmatrix} = -|A_0(t_0)\rangle_a$$

However, the states $|\underline{0,\varphi}\rangle_a$ and $|\underline{1,\varphi}\rangle_a$ are orthogonal, and thus must remain orthogonal in the course of a unitary evolution.

Let us analyze the evolution of the field part of the factorized wave function (5.34). This evolution is nonlinear and is governed by the effective field Hamiltonian,

$$H_\mathrm{eff} = \lambda_p g\sqrt{\hat{n}}$$

To better understand it, one may expand the square root $\sqrt{\hat{n}} = \sqrt{\overline{n} + \Delta\hat{n}}$ in powers of the operator $\Delta\hat{n}/\overline{n} \sim 1/\sqrt{\overline{n}}$:

$$H_\mathrm{eff} = \lambda_p g\sqrt{\hat{n}} \approx \lambda_p g\left(\sqrt{\overline{n}} + \frac{\Delta\hat{n}}{2\sqrt{\overline{n}}} - \frac{(\Delta\hat{n})^2}{8\overline{n}^{3/2}}\right) \tag{5.36}$$

Applying the linearized operator to the initial coherent state and neglecting the quadratic term $\sim 1/\sqrt{\overline{n}}$ for times $gt \ll \sqrt{\overline{n}}$, we have

$$|\Phi(t)\rangle_f = \mathrm{e}^{-\mathrm{i}\lambda_p gt\sqrt{\hat{n}}}|\alpha\rangle \approx \mathrm{e}^{-\mathrm{i}\lambda_p gt\sqrt{\overline{n}}/2}|\mathrm{e}^{-\mathrm{i}\omega_p t}\alpha\rangle \tag{5.37}$$

Here we have used the fact that $\mathrm{e}^{-\mathrm{i}\omega_p t\hat{n}}|\alpha\rangle = |\mathrm{e}^{-\mathrm{i}\omega_p t}\alpha\rangle$. This means that for short times (before the first revival), the initial coherent state simply rotates with the frequency ω_p in the phase plane of the field. For larger times, one has to take into account the anharmonic terms in Equation 5.36, which deform the shape of the initial coherent state.

It is important that the evolution of an arbitrary initial atomic state can be described in terms of the factorized states. To see this, we recall that the states $|\underline{p,\varphi}\rangle_a$ form a complete basis in the atomic Hilbert space, and the arbitrary initial atomic state can be written in this basis as

$$|\psi(0)\rangle_a = \sum_{p=0}^{1} C_p|\underline{p,\varphi}\rangle_a, \quad C_p = {}_a\langle\underline{p,\varphi}|\psi(0)\rangle_a$$

and thus, the evolution of this state in the presence of the initially coherent quantum field is given as a superposition of the factorized states

$$|\psi(t)\rangle = \sum_{p=0}^{1} C_p|A_p(t)\rangle_a|\Phi_p(t)\rangle_f \tag{5.38}$$

Let us take, for example, the atom in the initial ground state, $|\psi(0)\rangle_a = |0\rangle_a$, and the field in the coherent state with zero phase, $\alpha^2 = \overline{n}$, and calculate the atomic inversion evolution using the factorized wave functions (5.32). We have

$$\langle\psi(t)|\sigma_z|\psi(t)\rangle \approx \sum_{p,q=0}^{1} C_p C_q^* \,{}_a\langle A_q(t)|\sigma_z|A_p(t)\rangle_a \,{}_f\langle\Phi_q(t)|\Phi_p(t)\rangle_f \tag{5.39}$$

where $C_1 = -C_0 = 1/\sqrt{2}$. Using the definition of the atomic states (5.33), one can easily find the atomic matrix element

$${}_a\langle A_q(t)|\sigma_z|A_p(t)\rangle_a = {}_a\langle\underline{q}|\sigma_z \exp\left[-\mathrm{i}\frac{\omega_p-\omega_q}{2}(\sigma_z+1)\,t\right]|\underline{p}\rangle_a \tag{5.40}$$

In turn, the overlap integral between the field states

$${}_f\langle\Phi_q(t)|\Phi_p(t)\rangle_f = \sum_n P_n\,\mathrm{e}^{-\mathrm{i}(\lambda_p-\lambda_q)\,gt\sqrt{n}} \tag{5.41}$$

has the form of an anharmonic series. To get a qualitative image, one may use the linear approximation (Equation 5.37) for short times

$${}_f\langle\Phi_q(t)|\Phi_p(t)\rangle_f \approx \mathrm{e}^{-\mathrm{i}(\lambda_p-\lambda_q)\,gt\sqrt{\overline{n}}/2}\langle\mathrm{e}^{-\mathrm{i}\omega_q t}\alpha|\mathrm{e}^{-\mathrm{i}\omega_p t}\alpha\rangle \tag{5.42}$$

The rapidly oscillating factor $\mathrm{e}^{-\mathrm{i}(\lambda_p-\lambda_q)\,gt\sqrt{\bar{n}}/2}$ represents here the Rabi oscillations. Their slowly varying amplitude takes the form

$$|\langle \mathrm{e}^{-\mathrm{i}\omega_q t}\alpha|\mathrm{e}^{-\mathrm{i}\omega_p t}\alpha\rangle| = \exp\left[-\sqrt{\bar{n}}|\mathrm{e}^{-\mathrm{i}\omega_q t} - \mathrm{e}^{-\mathrm{i}\omega_p t}|^2/2\right] \tag{5.43}$$

The amplitude has its maximum at time $t = 0$ and the oscillations collapse when

$$\sqrt{\bar{n}}|\mathrm{e}^{-\mathrm{i}\omega_q t} - \mathrm{e}^{-\mathrm{i}\omega_p t}| \sim 1$$

Since $\omega_p \sim 1/\sqrt{\bar{n}}$ (and we are interested here in short times), we can expand the exponents in series and obtain the collapse time

$$T_{\mathrm{c}} = \frac{1}{\left(\omega_p - \omega_q\right)\sqrt{\bar{n}}}$$

On the other hand, it is seen from Equation 5.43 that the amplitude of the oscillations is once again $\sim$1 when

$$\left(\omega_p - \omega_q\right) T_{\mathrm{r}} = 2\pi k, \quad k = 1, 2, \ldots$$

which gives us the times of possible revivals. For these time instants, the atomic matrix element (Equation 5.40) is almost independent of time, since

$$\exp\left[-\mathrm{i}\frac{\omega_p - \omega_q}{2}\left(\sigma_z + 1\right) T_{\mathrm{r}}\right] = \exp\left[-\mathrm{i}\pi\left(\sigma_z + 1\right)\right] = 1$$

and therefore,

$${}_a\langle A_q(t)|\sigma_z|A_p(t)\rangle_a \approx {}_a\langle \underline{q}|\sigma_z|\underline{p}\rangle_a = \delta_{p,\,q\pm 1} \tag{5.44}$$

Finally, from Equation 5.39 using Equations 5.41 and 5.44 we find

$$\langle\psi(t)|\sigma_z|\psi(t)\rangle = -\sum_n P_n \cos(2\,gt\sqrt{n})$$

We thus recover the former expression for the atomic inversion.

5.9
Evolution in Field Phase Space

The factorized wave functions (5.32) lead to a very transparent dynamic picture in the field phase plane. To visualize the evolution of the field one may use, for example, the Q function, i.e. the matrix element of the field density matrix between the coherent states:

$$Q(\beta) = \langle\beta|\rho_f|\beta\rangle \tag{5.45}$$

Here, the coherent state parameter β takes arbitrary complex values and the complex β plane represents a classical phase space of the field. The field density matrix is obtained from the total density matrix taking the trace over the atomic variables, $\rho_f = \mathrm{Tr}_a\left(|\psi(t)\rangle\langle\psi(t)|\right)$. In the case of the semiclassical initial state, which leads to the factorized wave function, we have $\rho_f \approx |\Phi_p(t)\rangle_{f\,f}\langle\Phi_p(t)|$. Hence the Q function for a single semiclassical state is given by

$$Q(\beta) = |\langle\beta|\Phi_p(t)\rangle_f|^2$$

For short times $t < T_r$, when the linear approximation is valid, the Gaussian hump corresponding to the initial coherent state rotates around an origin of the phase plane along a circle of radius $\sqrt{\bar{n}}$ with the frequency ω_p. For larger times one cannot neglect the anharmonic terms in Equation 5.36. They deform the shape of the initial coherent state. The hump that corresponds to the state $|\Phi(t)\rangle_f$ acquires a typical crescent shape. The Q-function evolution at this stage can be described analytically [56, 57] taking into account the quadratic term in the expansion (5.36). The crescent shape leads to squeezing of the field quadrature components [57, 58]. Further on, the phase spread exceeds 2π, which leads to self-interference in the phase plane. This, in turn, leads to a kind of standing waves at some time instants, which are called fractional revivals [59]. This phenomena for the JCM can also be explained at the level of quadratic approximation to the frequencies. Note here that the field Hamiltonian

$$H = \omega\hat{n} + \chi\hat{n}^2 \tag{5.46}$$

has an independent physical sense and describes the optical Kerr medium (see Section 4.3) [34, 35]. This is the simplest model that reveals the phase spread due to the nonlinear evolution. It demonstrates, to some extent, dynamic features similar to those of the JCM [56]. For instance, the standing waves in the phase space are very well pronounced for a Kerr medium (it is the consequence of the integer spectrum of the nonlinear term $\hat{n}^2$). They are usually called the "Schrödinger cat states" [36].

For an arbitrary initial atomic state, the situation is more complicated. The initial coherent state is split into two humps, which correspond to the two different semiclassical states in the decomposition (5.38). These humps rotate with the frequencies $\omega_0 = -\omega_1 = g/2\sqrt{\bar{n}}$ (i.e. clockwise and counterclockwise) along the circle of radius $\sqrt{\bar{n}}$. When the humps are away from each other, there are no Rabi oscillations, since the overlap integral between the semiclassical states is zero, in accordance with Equations 5.39 and 5.42; this corresponds to the collapse region. From this, one can estimate once again the collapse time. Indeed, the Q function of a single semiclassical state is represented by a round hump of the radius ~ 1. The humps start their motion from the same point in opposite directions with linear velocities $|\omega_0|\sqrt{\bar{n}}$ and become well separated after a time $t \sim 2\omega_0\sqrt{\bar{n}} = 1/g = T_c$. The Rabi oscillations revive when the humps meet each other in the phase plane. This happens at the times $t = k\pi/\omega_0 = kT_r$. It is the interference of different semiclassical state that produces revivals of the Rabi oscillations.

5.10
The JCM without RWA

In this section, we consider the JCM without the RWA. Let us recall that the Hamiltonian has the form

$$H = H_0 + H_1 + H_2, \quad H_0 = \frac{1}{2}\,\omega_a\sigma_z + \omega_f a^\dagger a \tag{5.47}$$
$$H_1 = g\left(a\sigma_+ + a^\dagger\sigma_-\right), \quad H_2 = g\left(a^\dagger\sigma_+ + a\sigma_-\right)$$

where $H_0 + H_1$ is the JCM Hamiltonian under the RWA and H_2 represents the counter-rotating terms. (Note, that the counter-rotating terms in Equation 5.2 have a different sign; it can be changed by the appropriate unitary transformation.) In the classical field limit, these terms cause a small shift of the resonance frequency [22, 60], $\delta_{cl} = g^2\overline{n}/(\omega_f + \omega_a)$, where $\overline{n}$ is the field intensity in the units of the photon number.

We show now that the perturbation theory developed for the classical field (see Section 2.5) can be directly generalized to the quantum field, leading to a simple effective Hamiltonian that takes into account the influence of counter-rotating terms. In the quantum domain, the Bloch–Siegert shift depends on the photon number; the corresponding term in the effective Hamiltonian appears to be [61]

$$\frac{g^2}{2\underline{\omega}}\,\hat{n}\sigma_z, \quad \hat{n} \equiv a^\dagger a, \quad \underline{\omega} \equiv \frac{(\omega_a + \omega_f)}{2}$$

Terms of this kind are usually called *dynamical Stark shifts* [62]. The eigenvectors and eigenvalues of the effective Hamiltonian can be easily found. We finally apply this approach to find the atomic inversion evolution. It leads to a modification of the shape of the JCM collapses and revivals in the RWA.

5.10.1
Diagonalization of the Hamiltonian

We approximately transform the Hamiltonian (5.47) to one that commutes with the excitation- number operator $\hat{N}$, applying a sequence of small unitary transformations according to the general scheme described in Appendix 11.4. At the first step, we apply the following transformation to the Hamiltonian (5.47):

$$V = \exp\left[\epsilon A\right], \quad A = a^\dagger\sigma_+ - a\sigma_-$$

Assuming ϵ to be small, we neglect the terms of order ϵ^k, $k \geq 3$, writing

$$VHV^\dagger \approx H + \epsilon[A, H] + \frac{\epsilon^2}{2}\,[A, [A, H]]$$

The parts of the Hamiltonian are transformed as follows:

$$\begin{aligned} VH_0V^\dagger \approx\; & H_0 - 2\epsilon\underline{\omega}(a^\dagger\sigma_+ + a\sigma_-) \\ & - \epsilon^2\underline{\omega}[\sigma_z(2\hat{n}+1) - 1] \end{aligned} \tag{5.48}$$

$$\begin{aligned} VH_1V^\dagger \approx\; & H_1 + \epsilon g(a^{\dagger 2} + a^2)\sigma_z - \epsilon^2 g[(\sigma_+ a^{\dagger 3} + \sigma_- a^3) \\ & + (\sigma_+ a\hat{n} + \hat{n}a^\dagger\sigma_-)] \end{aligned} \tag{5.49}$$

$$\begin{aligned} VH_2V^\dagger \approx\; & H_2 + \epsilon g[\sigma_z(2\hat{n}+1) - 1] \\ & - 2\epsilon^2 g(\sigma_- a\hat{n} + \hat{n}a^\dagger\sigma_+) \end{aligned} \tag{5.50}$$

The second term in Equation 5.48 cancels the antirotating term H_2 provided we choose the parameter as follows:

$$\epsilon = \frac{g}{2\underline{\omega}}$$

Now, the zeroth order transformed Hamiltonian commutes with $\hat{N}$, and conserves the number of excitations. Let us note that the parameter ϵ is indeed small for a standard experimental setup. The third term in Equation 5.48 and the second one in Equation 5.50 are of the first order and give the quantum Bloch–Siegert shift.

In order to diagonalize the Hamiltonian up to the second order in ϵ in the basis of eigenstates of $\hat{N}$, we apply the second transformation

$$V_1 = \exp\left[\phi B\right], \; B = \left(a^{\dagger 2} - a^2\right)\sigma_z, \quad \phi = \frac{\epsilon g}{2\omega_f} \sim \epsilon^2$$

Then, the second term in the transformed H_0 Hamiltonian

$$V_1 H_0 V_1^\dagger \approx H_0 - \phi\, 2\omega(a^{\dagger 2} + a^2)\sigma_z - 8\phi^2\, \omega\left(\hat{n} + \frac{1}{2}\right)$$

cancels the second term in Equation 5.49, whereas the last term in the above equation is of order ϵ^3. The whole Hamiltonian after the transformations V and V_1 takes the following form:

$$\begin{aligned} V_1 V H V^\dagger V_1^\dagger \approx H_0 + \tilde{H}_1 + \varepsilon g\left[\sigma_z\left(\hat{n} + \frac{1}{2}\right) - \frac{1}{2}\right] - 2\epsilon^2\frac{\omega}{\omega_f} g\left[a^3\sigma_+ + a^{\dagger 3}\sigma_-\right] \\ + \epsilon^2 g\left[\frac{\omega_a}{\omega_f}\left(a^{\dagger 3}\sigma_+ + a^3\sigma_-\right) + \frac{\omega_a - \omega_f}{\omega_f}\left(\sigma_+\hat{n}a^\dagger + a\hat{n}\sigma_-\right)\right] \end{aligned} \quad (5.51)$$

where $\tilde{H}_1$ is the RWA interaction Hamiltonian with intensity-dependent coupling constant

$$\tilde{H}_1 = g(\hat{n})a^\dagger\sigma_- + \sigma_+ a g(\hat{n}), \quad g(\hat{n}) = g(1 - \epsilon^2\hat{n})$$

We notice that the last two terms in Equation 5.51 can be easily eliminated applying the corresponding transformations with rotation parameters proportional to ϵ^3 which, of course, does not affect the terms of the order of ϵ^2. To get rid of the term $\sim \epsilon^2 g(a^3\sigma_+ + a^{\dagger 3}\sigma_-)$ describing the three-photon transitions, one should apply the transformation

$$V_3 = \exp\left[\gamma\left(a^3\sigma_+ - a^{\dagger 3}\sigma_-\right)\right] \quad (5.52)$$

with

$$\gamma = 2\epsilon^2\frac{\omega}{\omega_f}\frac{g}{\omega_a - 3\omega_f} \quad (5.53)$$

If the atomic transition is far from the three-photon resonance, we get $\gamma \sim \epsilon^3$, and the term $\epsilon^2 g(a^3\sigma_+ + a^{\dagger 3}\sigma_-)$ can be eliminated. But in the resonant case, when $\omega_a \simeq 3\omega_f$, the parameter γ is great and the transformation (5.52) cannot be applied anymore. From Equation 5.53, it is easy to find the width of the three-photon resonance: $\omega_a - 3\omega_f \sim g\epsilon^2$, which is much smaller then the width of the first resonance, $\omega_a - \omega_f \sim g$. Let us note that in the three-photon resonance case the term describing the one-photon transitions, $g(a\sigma_+ + a^\dagger\sigma_-)$, can be removed by the appropriate small transformation.

To conclude, the effective Hamiltonian in the region close to the one-photon resonance takes the form

$$\begin{aligned} H_{\text{eff}} = {} & \frac{\omega_a}{2}\sigma_z + \omega_f a^\dagger a + \frac{g^2}{2\underline{\omega}}\left[\sigma_z\left(\hat{n} + \frac{1}{2}\right) - \frac{1}{2}\right] \\ & + g(\hat{n})a^\dagger\sigma_- + \sigma_+ a g(\hat{n}) \end{aligned} \tag{5.54}$$

where $g(\hat{n}) = g(1 - \epsilon^2\hat{n})$. It is clear that $[H_{\text{eff}}, \hat{N}] = 0$ and the dynamics with H_{eff} can be found exactly, as for the common JCM. Corrections produced by the counter-rotating terms appear in the first order as the intensity-dependent detuning ("dynamical Stark shift") and, in the second order, as the intensity-dependent coupling. One can rewrite the Stark shift term in Equation 5.54 as follows:

$$\epsilon g\left[\sigma_z\left(\hat{n} + \frac{1}{2}\right) - \frac{1}{2}\right] = \epsilon g\sigma_z\left(\hat{N} + \frac{1}{2}\right) - \epsilon g$$

This means that the atomic transition frequency acquires a different shift in each invariant subspace.

Let us denote the bare basis by $|N, k\rangle = |N - k\rangle_f |k\rangle_a$ (where $k = 0, 1$ is the atomic excitation number) and the dressed basis by $|\underline{N, k}\rangle$, so that the eigenvectors and eigenvalues of the initial Hamiltonian are

$$H|\underline{N, k}\rangle = \lambda_{N,k}|\underline{N, k}\rangle$$

The exact eigenvalues are approximated by those of the effective Hamiltonian (5.54), which are given by

$$\lambda_{N,k=0,1} = \omega N - \epsilon g + (-1)^k\Omega_N, \ \Omega_N = \sqrt{\frac{\Delta_N^2}{4} + g_N^2} \tag{5.55}$$

where

$$\Delta_N = \Delta + 2N\epsilon g, \quad g_N = g\sqrt{N}(1 - \epsilon^2 N), \quad \epsilon = \frac{g}{2\underline{\omega}}$$

Here the term $2N\epsilon g$ represents the dynamical Stark shift corresponding to the semiclassical Bloch–Siegert shift. The eigenvectors (dressed vectors) in the first order in ϵ take the following form:

$$\begin{aligned} |\underline{N, k}\rangle = {} & a_N^k|N, 0\rangle + b_N^k|N, 1\rangle \\ & + \epsilon\left[\sqrt{N-1}\, b_N^k|N-2, 0\rangle - \sqrt{N+1}\, a_N^k|N+2, 1\rangle\right] \end{aligned} \tag{5.56}$$

where

$$a_N^0 = b_N^1 = \sqrt{\frac{1}{2} - \frac{\Delta_N}{4\Omega_N}}, \quad a_N^1 = -b_N^0 = -\sqrt{\frac{1}{2} + \frac{\Delta_N}{4\Omega_N}}$$

In the zeroth order approximation, the two eigenvectors $|\underline{N,k}\rangle$, $k = 0, 1$ are localized in the subspace with N excitations, and in our notation, $k = 0$ corresponds to the higher eigenvalue (for a given N). Small contributions of the subspaces $N \pm 2$ appear in the first order.

5.10.2
Atomic Inversion

Let us assume that initially the field is taken in the coherent state with zero phase and the atom is prepared in its ground state

$$|\mathrm{in}\rangle = \sum P_n |n\rangle_f |0\rangle_a, \quad P_n = \sqrt{\frac{\mathrm{e}^{-\overline{n}}\overline{n}^n}{n!}}$$

The wave function evolution is given by

$$|\Psi(t)\rangle = \sum_{N,k} \exp(-\mathrm{i}t\lambda_{Nk}) C_{Nk} |\underline{N,k}\rangle$$

where C_{Nk} are coefficients of the expansion of the initial state in the dressed basis

$$C_{Nk} = \langle \underline{N,k} | \mathrm{in}\rangle = P_N a_N^k + \epsilon\sqrt{N-1} P_{N-2} b_N^k$$

In the first order approximation, we have

$$|\Psi(t)\rangle \approx |\Psi(t)\rangle^0 + \epsilon |\Psi(t)\rangle^1 \tag{5.57}$$

$$|\Psi(t)\rangle^0 = \sum_N \mathrm{e}^{-\mathrm{i}\omega N t} P_N \left\{ \left(\cos\Omega_N t + \frac{\mathrm{i}\Delta_N}{2\Omega_N} \sin\Omega_N t \right) |N,0\rangle \right. \\ \left. - \frac{ig_N}{\Omega_N} \sin\Omega_N t |N,1\rangle \right\}$$

$$|\Psi(t)\rangle^1 = \sum_N \mathrm{e}^{-\mathrm{i}\omega N t} \left\{ \left[-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}2\omega t} \sin\Omega_{N+2}\sqrt{N+1} P_{N+2} \frac{g_{N+2}}{\Omega_{N+2}} \right. \right. \\ \left. - \mathrm{i}\sin\Omega_N t \sqrt{N-1} P_{N-2} \frac{g_N}{\Omega_N} \right] |N,0\rangle \\ + \sqrt{N-1} P_{N-2} \left[\mathrm{e}^{\mathrm{i}2\omega t} \left(-\cos\Omega_{N-2} t - \frac{\mathrm{i}\Delta_{N-2}}{2\Omega_{N-2}} \sin\Omega_{N-2} t \right) \right. \\ \left. \left. + \cos\Omega_N t - \frac{\mathrm{i}\Delta_N}{2\Omega_N} \sin\Omega_N t \right] |N,1\rangle \right\}$$

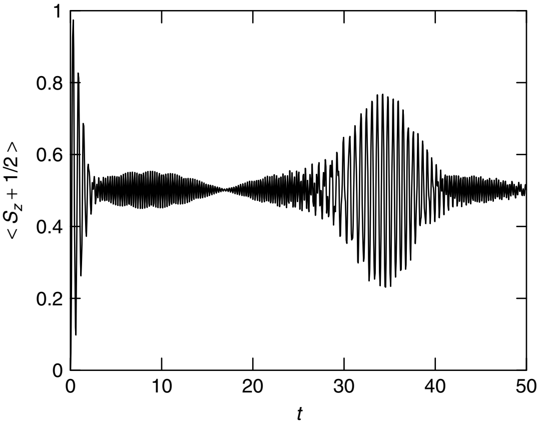

Figure 5.3 Dynamics of atomic inversion in the JCM without RWA for the initial coherent state of the field, $\overline{n} = 30$, and the ground state of the atom; $g = 1$, $\omega = \omega_a = 50$.

The atomic inversion $\langle \Psi(t)|\sigma_z|\Psi(t)\rangle$ versus the time ($g = 1$) is shown in Figure 5.3 for the values $\omega_a = \omega_f = 50$ and $\overline{n} = 30$. This picture shows the results of exact numerical diagonalization of the Hamiltonian (5.47); the calculation with the wave function (5.57) is in an excellent agreement with the exact results. The small parameter $\epsilon = g/(\omega_a + \omega_f) = 0.01$. The graph differs significantly form the atomic inversion picture for the JCM under RWA, Figure 5.1. For instance, the rapid oscillations with the frequency $\sim \omega$ are still present when the Rabi oscillations (frequency $\sim \Omega_{\overline{n}}$) are collapsed.

5.10.3
Classical Field Limit

In the rotating frame (making the transformation $\exp(i\omega_f \hat{n} t)$), the Hamiltonian (5.47) takes the form

$$H(t) = \frac{1}{2}\,\omega_a\sigma_z + g\sigma_x\left(a\,e^{-i\omega_f t} + a^{\dagger}\,e^{i\omega_f t}\right) \tag{5.58}$$

Usually, the classical field limit is obtained by replacing here the field operators by c-numbers: $a \to \alpha$. Then

$$H_{cl}(t) = \frac{1}{2}\,\omega_a\sigma_z + g\sigma_x\left(\alpha\,e^{-i\omega_f t} + \alpha^* e^{i\omega_f t}\right)$$

which is the Hamiltonian studied by Shirley [22]. However, the quasi-classical limit can be obtained in a different way. Let us take the field initially in a coherent state. Then the relative fluctuations of the field intensity become small $\Delta n \ll \overline{n}$, and one may substitute $a \to \sqrt{\overline{n}}\exp(i\hat{\phi})$ in the initial Hamiltonian (5.47), where $\overline{n}$ is the field intensity and $\exp(i\hat{\phi})$ is the phase operator $(\exp i\hat{\phi}|n\rangle = |n-1\rangle)$. (Note, that $[\exp(i\hat{\phi}), \exp(-i\hat{\phi})] = 0$, if the contribution of the vacuum state is neglected.) Then the Hamiltonian (5.47) takes the form

$$H_{\text{cl}} = \omega_f \hat{n} + \frac{1}{2}\,\omega_{\text{a}}\sigma_z + g\sqrt{\overline{n}}\sigma_x\left(\text{e}^{\text{i}\hat{\phi}} + \text{e}^{-\text{i}\hat{\phi}}\right) + O(1) \tag{5.59}$$

which coincides with the Hamiltonian (2.19) used in Section 2.5. Correspondingly, its approximate eigenvalues and eigenvectors have the forms (5.55) and (5.56), where N is replaced by $\overline{n}$. In particular, the dynamical Stark shift $2N\epsilon g$ is transformed into the Bloch–Siegert shift.

In the rotating frame, the Hamiltonian (5.59) can be rewritten as

$$H_{\text{cl}}(t) = \frac{1}{2}\,\omega_{\text{a}}\sigma_z + g\sqrt{\overline{n}}\sigma_x\left(\text{e}^{\text{i}(\hat{\phi}-\omega_f t)} + \text{e}^{-\text{i}(\hat{\phi}-\omega_f t)}\right)$$

It is clear now that the field operators $\exp\left(\pm\text{i}\hat{\phi}\right)$ commute with the above Hamiltonian. In this approximation, the field variables are still considered as operators; however, the atomic dynamics does not affect the field and the total wave function is factorized into the atomic and field parts. The semiclassical Hamiltonian (5.59) describes well only the short-time region $gt < 1$, which covers times before the first collapse.

5.11
Problems

5.1 Find how the atomic purity $P = \text{Tr}(\rho_{\text{at}}^2)$ evolves in the resonance JCM for the following initial states: (a) the field is in the vacuum state and the atom is excited; (b) the field is in a strong coherent state and the atom is in the ground state.

5.2 Find the evolution of the average values of the Heisenberg operators that describe the atomic dipole momentum, $\sigma_x(t), \sigma_y(t)$ and their fluctuations in the JCM for the initial state with the field in a strong coherent state and the atom in a semiclassical state.

5.3 Find the Heisenberg operators $\sigma_x(t), \sigma_z(t)$, and $a(t)$ in the JCM using the evolution operator in the form (5.14).

5.4 Write the system of Heisenberg equations for the operators $X_+ = a\sigma_+$, $X_- = a^\dagger\sigma_-$, σ_z. Reduce this system to a single equation for σ_z and solve it.

5.5 Prove that the operators

$$L_+ = \frac{1}{\sqrt{N}}a\sigma_+, \quad L_- = \frac{1}{\sqrt{N}}a^\dagger\sigma_-, \quad L_z = \sigma_z$$

where

$$N = a^\dagger a + \frac{1}{2}\sigma_z + \frac{1}{2}$$

form a representation of the *su*(2) algebra in a space with a fixed number of excitations.

5.6 Write the Heisenberg equations for the operators σ_-, σ_z, a. Reduce this system of differential equations for the operators to a system of equations for average values by uncoupling correlators: $\langle \hat{f}\hat{g}\rangle \to \langle \hat{f}\rangle\langle \hat{g}\rangle$. Using the change of variables,

$$\langle a\rangle = \alpha\, e^{i\varphi}, \quad \langle \sigma_-\rangle = s\sin\theta\, e^{i(\varphi-\chi)}, \qquad \langle \sigma_z\rangle = 2s\cos\theta$$

find integrals of motion for this system of equations. (In the above equations s is one of the integrals of motion.) Reduce the system to two equations for θ and χ, and solve them in a strong field limit: $\alpha(t) \gg s$. Show that such a solution works well only in the case of initial semiclassical states (5.31).

5.7 Find the spectrum and the evolution operator for the Hamiltonian that describes the JCM with k photon transitions:

$$H = \omega a^\dagger a + \frac{1}{2}\omega_0\sigma_z + g\left(a^{\dagger k}\sigma_- + a^k\sigma_+\right)$$

Analyze the existence of trapping states.

5.8 For factorized states in the JCM analyze the effect of field squeezing.

5.9 Find the evolution of the field phase and phase fluctuations for the JCM if the atom is initially taken in a semiclassical state $|p\rangle_a$ and the field in a number state $|N\rangle_f$, $N \gg 1$.

5.10 Find the spectrum of the Hamiltonian that describes the interaction of a two-level atom with a single mode of the quantized field in the presence of the Kerr medium,

$$H = \omega a^\dagger a + \frac{1}{2}\omega_0\sigma_z + g\left(a^\dagger\sigma_- + a\sigma_+\right) + \chi\left(a^\dagger a\right)^2$$

What is the principal difference from the standard JCM?

6
Collective Interactions

6.1
The Dicke Model (Exactly Solvable Examples)

Here we consider a collection of A identical, but, in principle, distinguishable two-level atoms placed in an ideal cavity and interacting with an electromagnetic field cavity mode with a frequency ω_f nearly at resonance with the atomic transition frequency ω_a. The following standard approximations are applied:

1. All the atoms are assumed to be localized within a volume that is small compared to the field-mode wavelength λ; this means that all the atoms "feel" the same electromagnetic field, and the coupling constant g (which is expressed in terms of the matrix element of the atomic dipole momentum operator between the corresponding states) is the same for all the atoms.
2. The wave functions of different atoms do not overlap, which means that the dipole–dipole interactions between the atoms are neglected.
3. The atoms are assumed to be immobile, i.e. the Doppler effect and the consequences of interatomic collisions are not taken into account.
4. The interaction with the field is described in the framework of the rotating wave approximation, which consists of discarding terms which violate the energy-conservation law in first-order perturbation theory.

A system of this kind was considered for the first time in 1954, in the well-known paper by Dicke [13], and we call it the *Dicke model* (though the term "the Tavis–Cummings model" is also used) [63]. The Hamiltonian has the form ($\hbar = 1$)

$$H = \omega_f a^\dagger a + \frac{\omega_a}{2}\sum_{j=1}^{A} \sigma_3^j + g\sum_{j=1}^{A}\left(a^\dagger \sigma_-^{(j)} + a\sigma_+^{(j)}\right) \tag{6.1}$$

where $a^\dagger$ and a are the field creation and annihilation operators ($[a, a^\dagger] = 1$); $\sigma_\pm^{(j)}$, $\sigma_3^{(j)}$ are Pauli matrices describing transitions between resonance levels of the jth atom and the effective Hamiltonian of the isolated jth atom. The important property of the Hamiltonian (6.1) is its symmetry with respect to permutations of atoms. As a result, the symmetry properties of the initial state with respect to atomic

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permutations is preserved during the entire evolution. In particular, if the initial state is fully symmetric, the motion takes place only over symmetric states, a fact connected with the well-known superradiance effect [13].

Another essential property of the Hamiltonian (6.1) is the existence of the integral of motion that commutes with the Hamiltonian:

$$\hat{N} = a^{\dagger} a + \frac{1}{2} \sum_{j=1}^{A} \sigma_3^j + \frac{A}{2} \tag{6.2}$$

N has a sense of a number of excitations in the system. Owing to the existence of the integral of motion, this problem belongs to the class of "exactly solvable", in the sense that the Hamiltonian has a block diagonal structure where all the blocks (invariant subspaces) are finite dimensional. Under certain initial conditions, the invariant subspaces have low dimensions and the dynamic problem can be solved exactly without the use of perturbation theory or other approximate methods.

We start by considering some particular cases of initial states, when there is only one excitation in the system, which means that the evolution takes place in a subspace corresponding to the eigenvalue $N = 1$ of the excitation-number operator. For these cases, one can easily find exact solutions, which demonstrate that the dynamics is strongly affected by the symmetry properties of the initial state.

If one excited atom and no photons exist in the initial state, then the wave function of the whole system has the form

$$\left|\Psi(t)\right\rangle = C^{(1)}(t)\,|1\rangle + \sum_{l=1}^{A} C_l^{(2)}(t)\,|2_l\rangle \tag{6.3}$$

where $|1\rangle = |0_1, 0_2, \ldots, 0_N\rangle_{\rm a} \otimes |1\rangle_f$ denotes a state where all the atoms are not excited and one photon is present. The vectors $|2_l\rangle = |0_1, 0_2, \ldots, 1_l, \ldots 0_N\rangle_{\rm a} \otimes |0\rangle_f$ correspond to states where the lth atom is excited, the remaining atoms are in the ground state, and there are no photons; $C^{(1)}$ and $C_l^{(2)}$ are the probability amplitudes of the states $|1\rangle$ and $|2_l\rangle$. In examples 1–4 that follow we assume exact resonance: $\omega_f = \omega_{\rm a}$.

1. *Symmetric initial conditions.* We suppose that initial state of the atoms in the cavity was prepared by exciting the atoms by a single photon, so that

$$\left|\Psi(t=0)\right\rangle = \frac{1}{\sqrt{A}} \sum_{j=1}^{A} \left|2_j\right\rangle$$

i.e. the initial state is a superposition that includes all states in which one of the A atoms is excited. The probability that lth atom is excited is equal to A^{-1}. In this case, both the Hamiltonian (6.1) and the initial state are symmetric with respect to permutations of the atoms, and this symmetry is conserved for all times: $C_1^{(2)}(t) = \cdots = C_A^{(2)}(t) \equiv C^{(2)}(t)$. The nonstationary Schrödinger equation reduces to a system of two equations

$$\mathrm{i}\dot{C}^{(1)} = gAC^{(2)}, \quad \mathrm{i}\dot{C}^{(2)} = gC^{(1)} \tag{6.4}$$

with the initial conditions

$$C^{(1)}(t=0)=0, \quad C^{(2)}(t=0)=\frac{1}{\sqrt{A}}$$

which can be solved directly. The exact probability amplitudes are given by

$$C^{(1)}(t)=-\mathrm{i}\sin\Omega_0 t, \quad C^{(2)}(t)=\frac{1}{\sqrt{A}}\cos\Omega_0 t, \quad \Omega_0=g\sqrt{A}$$

The probability that all the atoms are in their ground states is equal to

$$P_1(t)=|C^{(1)}(t)|^2=\sin^2(g\sqrt{A}t)$$

while the probability that one of the atoms gets excited is given by

$$P_2(t)=A|C^{(2)}(t)|^2=\cos^2(g\sqrt{A}t), \quad P_1(t)+P_2(t)=1$$

2. *Asymmetric initial excitation.* Assume that the excited atom enters the cavity at the time instant $t=0$, while the cavity contains $A-1$ unexcited atoms of the same type. From the symmetry with respect to permutation of $A-1$ unexcited atoms, we have $C_2^{(2)}=C_3^{(2)}=\cdots=C_A^{(2)}$. The Schrödinger equation for the wave function (6.3) reduces now to a system of three equations for the amplitudes $C^{(1)}$, $C_1^{(2)}$ and $C_l^{(A)}$, $2\le l\le A$:

$$\begin{aligned}\mathrm{i}\dot{C}^{(1)}=gC_1^{(2)}+g(A-1)C_l^{(2)}, &\quad C^{(1)}(t=0)=0\\ \mathrm{i}\dot{C}_1^{(2)}=gC_1, &\quad C_1^{(2)}(t=0)=1\\ \mathrm{i}\dot{C}_l^{(2)}=gC_1, &\quad C_l^{(2)}(t=0)=0\end{aligned}$$

The solution is of the form

$$\begin{aligned}C^{(1)}(t)&=-\frac{\mathrm{i}}{\sqrt{A}}\sin\Omega_0 t\\ C_1^{(2)}(t)&=\frac{1+(\cos\Omega_0 t-1)}{A}\\ C_l^{(2)}(t)&=\frac{(\cos\Omega_0 t-1)}{A}, \quad \Omega_0=g\sqrt{A}\end{aligned} \tag{6.5}$$

where the Rabi frequency Ω_0 is the same as for the symmetric case. The probability that the first atom remains excited is

$$P_1^{(2)}(t)=|C_1^{(2)}(t)|^2=\left[\frac{1+(\cos\Omega_0 t-1)}{A}\right]^2$$

the probability that one among the rest of the atoms will be excited is

$$P_l^{(2)}(t)=|C_l^{(2)}(t)|^2(A-1)=\frac{A-1}{A^2}(\cos\Omega_0 t-1)^2$$

the probability P_0 that there are no photons in the cavity, and the probability P_1 that one photon is present are equal to

$$P_0 = P_1^{(2)} + P_l^{(2)} = 1 - \frac{1}{A}\sin^2(g\sqrt{A}t)$$

$$P_1 = |C^{(1)}(t)|^2 = \frac{1}{A}\sin^2(g\sqrt{A}t)$$

If A becomes very large, we have

$$P_1^{(2)} \to 1, \quad P_l^{(2)} \sim \frac{1}{A} \to 0\ (l \neq 1), \quad P_1 \sim \frac{1}{A} \to 0$$

This means that the initially excited atom will never emit photons. The presence of $A - 1$ unexcited atoms in the cavity prevents emission by the injected excited atom. This phenomenon is named ''radiation trapping'' [64].

3. *One photon and A unexcited atoms at the initial instant of time*. The evolution is over the symmetric states with respect to atomic permutations, in analogy with case 1. Solving the equation of motion (6.4) with initial conditions $C^{(1)}(0) = 1$, $C^{(2)}(0) = 0$, we obtain

$$C^{(1)} = \cos\Omega_0 t, \quad C^{(2)} = \frac{-\mathrm{i}\sin\Omega_0 t}{\sqrt{A}}, \quad \Omega_0 = g\sqrt{A}$$

For the probabilities P_1 that one photon is present and P_0 that there are no photons, we have

$$P_1(t) = |C^{(1)}|^2 = \cos^2 g\sqrt{A}t, \quad P_1(t) = A|C^{(2)}|^2 = \sin^2 g\sqrt{A}t$$

The evolution thus differs from the case 1 only by a time shift.

4. *Asymmetric initial condition without the "small-volume approximation"*. Now we consider the case when the coupling constant g depends on the position of the atom; this occurs if the sample dimensions are comparable with the wavelength. We will see that the radiation trapping is still present for the asymmetric initial condition. The interaction Hamiltonian (in the exact resonance case) takes the form

$$H = \sum_{j=1}^{A} g_j(a\sigma_+^{(j)} + a^\dagger\sigma_-^{(j)}) \tag{6.6}$$

The evolution of the asymmetric initial state $\left|\Psi(t=0)\right\rangle = |2_1\rangle$ with the first atom excited again takes place inside a subspace corresponding to the eigenvalue $N = 1$ of the excitation-number operator (6.2). The Hamiltonian (6.6) acts in the basis (6.3) as follows:

$$H|1\rangle = \sum_{j=1}^{A} g_j\left|2_j\right\rangle, \quad H\left|2_j\right\rangle = g_j|1\rangle$$

so that

$$H^2 |1\rangle = \sum_{j=1}^{A} g_j^2 |1\rangle , \quad H^2 \left|2_j\right\rangle = g_j \sum_{l=1}^{A} g_l |2_l\rangle$$

and thus,

$$H^3 |1\rangle = \Lambda^2 H |1\rangle , \quad H^3 \left|2_j\right\rangle = \Lambda^2 H \left|2_j\right\rangle$$

where $\Lambda^2 = \sum_{l=1}^{A} g_l^2$. This allows us to find the evolution operator in the given subspace in a simple form

$$\exp(-\mathrm{i}Ht) = 1 + H^2 \left(\frac{\cos \Lambda t - 1}{\Lambda^2} \right) - \mathrm{i}H \frac{\sin \Lambda t}{\Lambda}$$

The wave function has the form,

$$\left|\Psi(t)\right\rangle = \exp(-\mathrm{i}Ht) |2_1\rangle = C^{(1)}(t) |1\rangle + \sum_{l=1}^{A} C_l^{(2)} |2_l\rangle$$

where the amplitudes $C^{(1)}$ and $C_l^{(2)}$ are now given by the expressions

$$C^{(1)}(t) = -\mathrm{i}g_1 \frac{\sin \Lambda t}{\Lambda}, \quad C_1^{(2)}(t) = 1 + g_1^2 \frac{\cos \Lambda t - 1}{\Lambda^2} \tag{6.7}$$
$$C_l^{(2)}(t) = g_1 g_l \frac{\cos \Lambda t - 1}{\Lambda^2}, \quad 2 \le l \le A$$

Note that in the case $g_1 = g_2 = \cdots = g_A = g$, we have $\Lambda = g\sqrt{A} = \Omega_0$ and recover the result of the example 2. In a general case, we can introduce an effective coupling constant

$$g_{\text{ef}}^2 = \frac{\Lambda^2}{A} = \frac{1}{A} \sum_{j=1}^{A} g_j^2$$

If g_{ef} remains finite as $A \to \infty$, we obtain from Equation 6.7 in the limit of a large number of atoms

$$C^{(1)}(t) = -\mathrm{i}\frac{g_1}{g_{\text{ef}}} \frac{\sin \Lambda t}{\sqrt{A}} \sim \frac{1}{\sqrt{A}} \to 0$$
$$C_1^{(2)}(t) = 1 + \left(\frac{g_1}{g_{\text{ef}}} \right)^2 \frac{\cos \Lambda t - 1}{A} \to 1$$
$$C_l^{(2)}(t) = \frac{g_1 g_l}{g_{\text{ef}}^2} \frac{\cos \Lambda t - 1}{A} \sim \frac{1}{\sqrt{A}} \to 0$$

Thus, for a large numbers of atoms $A \to \infty$, the probabilities of the states are similar to those in the asymmetric case (6.5), so that radiation trapping is observed here, too.

6.2
The Dicke Model (Symmetry Properties)

The above-mentioned symmetry under atomic permutations leads to a very specific structure of the Hilbert space for the Dicke model. We start with the analysis of the Hilbert space of the system of A two-level atoms [65]. We denote by $\hat{p}$ an arbitrary permutation of atoms. As we understand, the collective atomic operators commute with $\hat{p}$:

$$S_z = \frac{1}{2}\sum_j \sigma_{zj}, \quad S_\pm = \sum_j \sigma_{\pm j}, \quad [S_{\pm,z},\ \hat{p}] = 0$$

In this section, we also use the standard notations from the angular momentum theory, where the state of angular momentum $j = A/2$ with $k = A/2 - m$ excitations is denoted by

$$\begin{aligned} &\left|j,\ m\right\rangle,\ S_z\left|j,\ m\right\rangle = m\left|j,\ m\right\rangle,\ -\frac{A}{2} \le m \le \frac{A}{2} \\ &S_\pm\left|j,\ m\right\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\left|j,\ m \pm 1\right\rangle \end{aligned}$$

The ground state

$$|0\rangle_{\mathrm{at}} = |0_1, 0_2, \ldots, 0_A\rangle = \left|\frac{A}{2},\quad -\frac{A}{2}\right\rangle$$

is nondegenerate. It is annihilated by the atomic lowering operator S_-:

$$S_-\left|\frac{A}{2},\quad -\frac{A}{2}\right\rangle = 0$$

and thus it is the lowest state (“vacuum vector”) of the $(A+1)$-dimensional irreducible representation of the $su(2)$ algebra. The basis of this representation is defined as follows:

$$\left|\frac{A}{2},\quad -\frac{A}{2} + k\right\rangle = \mathcal{A}_k S_+^k \left|\frac{A}{2},\quad -\frac{A}{2}\right\rangle, \quad k = 0, 1, \ldots, A$$

where $\mathcal{A}_k$ are the normalization coefficients and $S_+^{A+1}|\frac{A}{2}, -\frac{A}{2}\rangle = 0$.

The first excited state is A-fold degenerate: we have a single fully symmetric state

$$\begin{aligned} |1, 0\rangle_{\mathrm{at}} &= \left|\frac{A}{2},\quad -\frac{A}{2} + 1\right\rangle = \mathcal{A}_1 S_+ \left|\frac{A}{2},\quad -\frac{A}{2}\right\rangle \\ &= \frac{|1_1, 0_2, \ldots, 0_A\rangle + |0_1, 1_2, \ldots, 0_A\rangle + \cdots + |0_1, 0_2, \ldots, 1_A\rangle}{\sqrt{A}} \end{aligned}$$

and $A - 1$ nonsymmetric states that correspond to the same number of atomic excitations. They can be chosen in different ways; for instance, as eigenstates of a cyclic atomic permutation

$$\hat{p}_c = \begin{pmatrix} 1 & 2 & 3 & \cdots & A-1 & A \\ 2 & 3 & 4 & \cdots & A & 1 \end{pmatrix} \tag{6.8}$$

$$|1, \alpha\rangle_{\text{at}} = \frac{1}{\sqrt{A}} \sum_{k=1}^{A} w^{k\alpha} |0_1, \ldots, 1_k, \ldots 0_A\rangle, \quad \alpha = 1, 2, \ldots, A-1$$

where $w = e^{i2\pi/A}$. The states $|1, \ \alpha\rangle_{\text{at}}$, $\alpha = 0, \ 1, \ \ldots, A-1$ form an orthonormal basis in the subspace with one atomic excitation. One can check that these states (for $\alpha > 0$) are annihilated by the lowering operator S_-:

$$S_- |1, \ \alpha\rangle_{\text{at}} = 0, \quad \alpha = 1, 2, \ldots, A-1$$

and that they are vacuum vectors of different $(A-1)$-dimensional irreducible representations of the $su(2)$ algebra. Using the angular momentum notation,

$$|1, \ \alpha\rangle_{\text{at}} = \left| \frac{A}{2} - 1, \quad -\frac{A}{2} + 1, \ \alpha \right\rangle$$

we write the basis of these representations as

$$\left| \frac{A}{2} - 1, \quad \frac{A}{2} + 1 + k, \ \alpha \right\rangle = \mathcal{A}_k S_+^k \left| \frac{A}{2} - 1, \quad -\frac{A}{2} + 1, \quad \alpha \right\rangle,$$
$$k = 0, 1, \ldots, A-2$$

Note, that the symmetry properties of all the basis vectors of the representation with given α are the same, due to the fact that $[S_+, \hat{p}] = 0$.

The second excited state is $A(A-1)/2$-fold degenerate: we have one state from the fully symmetric representation,

$$\left| \frac{A}{2}, \quad -\frac{A}{2} + 2 \right\rangle = \sqrt{\frac{2}{A(A-1)}} \sum_{j<k} |\ldots, 1_j, \ldots, 1_k, \ldots\rangle$$

$A-1$ states from the $(A-1)$-dimensional representations,

$$\left| \frac{A}{2} - 1, \quad -\frac{A}{2} + 2, \quad \alpha \right\rangle = \mathcal{A}_1 S_+ \left| \frac{A}{2} - 1, \quad -\frac{A}{2} + 1, \alpha \right\rangle$$
$$\alpha = 1, 2, \ldots, A-1$$

and $d_2 = A(A-1)/2 - A$ more states, which can be chosen to be orthogonal. They are the vacuum states of the $(A-3)$-dimensional irreducible $su(2)$ representations:

$$S_- \left| \frac{A}{2} - 2, \quad -\frac{A}{2} + 2, \quad \alpha \right\rangle = 0, \quad \alpha = 1, 2, \ldots, d_2$$

where the index α enumerates these new representations.

The k–th excited state ($k \le A/2$) has $A!/k!(A-k)!$-fold degeneracy. Of these, $A!/(k-1)!(A-k+1)!$ states belong to the $su(2)$ representations that have appeared before, and there are

$$d_k = \frac{A!}{(k-1)!(A-k+1)!} - \frac{A!}{k!(A-k)!} \tag{6.9}$$

new states, which can be chosen to be orthogonal,

$$\left|\frac{A}{2} - k,\ -\frac{A}{2} + k,\ \alpha\right\rangle, \quad \alpha = 1, 2, \ldots d_k \tag{6.10}$$

They are the lowest states of the $(A+1-2k)$ dimensional irreducible $su(2)$ representations. Therefore, the Hilbert space of the atomic system is a direct sum of the irreducible representations of the $su(2)$ algebra, in which the $(A+1-2k)$ dimensional representation appears d_k times (Equation 6.9).

Now we can better understand the phenomenon of radiation trapping, which appeared in the previous section. In accordance with the discussion above, the state $|j, -j, \alpha\rangle$ has $A/2 - j$ atomic excitations, while the state $\left|j, -j+k, \alpha\right\rangle$ corresponds to $A/2 - j + k$ atomic excitations. The initial atomic state with N excitations can be written as a sum of the states belonging to different $su(2)$ representations:

$$|\text{in}\rangle_{\text{at}} = \sum_{j=A/2-N}^{A/2} \sum_{\alpha} C_{j\alpha} \left|j, N - \frac{A}{2},\ \alpha\right\rangle \tag{6.11}$$

Due to conservation of symmetry of the initial state, transitions are possible only between states of the same irreducible $su(2)$ representations. In turn, among the different representations that appear in the expansion (6.11), only the fully symmetric representation $j = A/2$ permits transitions to the ground state. All the other representations lead to radiation trapping [67].

We consider the nonsymmetric initial atomic state with the first N atoms in the excited state and the rest in their nonexcited states:

$$|1_1, 1_2, \ldots, 1_N, 0_{N+1}, \ldots, 0_A\rangle\,, \quad K \equiv A - N$$

We assume that $A \gg N$. To estimate the probability of the absence of transitions from this state to the lower states (trapping), we note that this state has a special symmetry under permutations of the first N atoms and under permutations of the last K atoms. As a result, the expansion (6.11) is simplified. One can write

$$|1_1, 1_2, \ldots, 1_N, 0_{N+1}, \ldots, 0_A\rangle = \left|\frac{N}{2}, \frac{N}{2}\right\rangle \otimes \left|\frac{K}{2}, -\frac{K}{2}\right\rangle$$

where

$$|1_1, 1_2, \ldots, 1_N\rangle = \left|\frac{N}{2},\ \frac{N}{2}\right\rangle,\ |0_{N+1}, 0_{N+2}, \ldots, 0_A\rangle = \left|\frac{K}{2},\ -\frac{K}{2}\right\rangle$$

Therefore, the initial states belong to the tensor product of two irreducible $su(2)$ representations, of dimensions $(N+1)$ and $(K+1)$. This tensor product can be decomposed into irreducible components as follows ($l > j$):

$$\left|j,\ m\right\rangle \otimes \left|l,\ n\right\rangle = \sum_{J=l-j}^{l+j} C_{jm,ln}^{J,m+n} \left|J, m+n\right\rangle$$

where $C_{jm,ln}^{J,m+n}$ are the Clebsch–Gordan coefficients. For our initial condition, the Clebsch–Gordan coefficients have a very simple form (see, e.g. [68]):

$$\left|\frac{N}{2}, \frac{N}{2}\right\rangle \otimes \left|\frac{K}{2},\ -\frac{K}{2}\right\rangle = \sum_{J=(K-N)/2}^{(K+N)/2} C_{(N/2N)/(2),\ (K/2-K/2)}^{J,\ (N-K)/2} \left|J, \frac{N+K}{2}\right\rangle$$

$$C_{aa,b-b}^{c,a-b} = \left[\frac{(2c+1)(2a)!(2b)!}{(a+b+c+1)!(a+b-c)!}\right]^{1/2}$$

We see that the contribution of the representation with $J = (K-N)/2$ is the greatest:

$$C_{(N/2)(N/2),(K/2-K/2)}^{(K-N)/2,(N-K)/2} \sim \sqrt{\frac{K-N+1}{K+1}}$$

The probability that with the initial state, the system finds itself in this representation is given by

$$P\left(J = \frac{(K-N)}{2}\right) = \left|C_{(N/2)\,(N/2),\,(K/2-K/2)}^{(K-N)/2,(N-K)/2}\right|^2 \sim \frac{K-N}{K}$$

The probability of the next representation, $J = (K-N)/2 + 1$ is suppressed by the factor N/K. On the other hand, the lowest state in the representation $J = (K-N)/2$ corresponds to N excitations. Thus transitions to lower excitation numbers are impossible in this representation and we are faced with radiation trapping.

6.3
The Dicke Model (Symmetric Case)

Now, we turn to the fully symmetric case when the initial atomic state is invariant under all permutation of atoms. The interaction Hamiltonian (6.1) can be written in terms of collective atomic operators (1.11) as follows:

$$H = H_0 + V, \quad H_0 = \omega a^{\dagger} a + \omega_{\mathrm{a}} S_z \tag{6.12}$$

$$V = g\left[a^{\dagger} S_- + a S_+\right] \tag{6.13}$$

Recall that the operators $S_{\pm,z}$ generate the $(A+1)$-dimensional representation of the $su(2)$ algebra obeying the standard commutation relations, $[S_z, S_\pm] = \pm S_\pm$, $[S_+, S_-] = 2S_z$, so that their action in the basis of the Dicke states $|A, k\rangle_{\mathrm{a}}$, $k = 0, 1, \dots A$, where k is the number of symmetric atomic excitations, is defined by Equations 1.12. It is convenient to use the Schwinger realization for the atomic operators in terms of two fictitious boson modes, $[b_i, b_j^\dagger] = \delta_{ij}$, $i, j = 1, 2$,

$$S_z = \frac{(b_2^\dagger b_2 - b_1^\dagger b_1)}{2}, \quad S_+ = b_1 b_2^\dagger, \quad S_- = b_1^\dagger b_2$$

The Dicke states are then written in terms of Fock states of the modes $|k\rangle_{1,2}$ as

$$|k\rangle_a = |A - k\rangle_1 |k\rangle_2$$

The interaction Hamiltonian (6.13) takes a trilinear form

$$V = a\, b_1 b_2^\dagger + a^\dagger b_1^\dagger b_2 \tag{6.14}$$

The number of atoms A and the excitation number N are written as

$$A = b_1^\dagger b_1 + b_2^\dagger b_2, \quad \hat{N} = a^\dagger a + b_2^\dagger b_2$$

and they are integrals of motion for the Hamiltonian (6.14). The three-photon Hamiltonian (6.14) is interesting by itself, describing such optical processes as frequency conversion, and Raman and Brillouin scattering [69]). The operators a, b_1, and b_2 are usually called *signal, idler* and *pump* modes. From the mathematical point of view the Hamiltonians (6.13) and (6.14) are fully equivalent.

For simplicity, we restrict ourselves to the exact resonance case $\omega_{\mathrm{a}} = \omega$. Then the integral of motion corresponding to the excitation-number operator $\hat{N} = \hat{n} + S_z$ is proportional to the bare Hamiltonian H_0.

6.4
The Zeroth-Order Approximation

6.4.1
The Weak Field Case

The two cases $A > N$ and $A < N$ are essentially different and are treated separately. First, we assume that the number of photons in the cavity is small comparing with the number of atoms, and the atoms are initially in their ground states; thus $A > N$. The initial excitation number is conserved, and only the states

$$|N, n\rangle \equiv |A, N - n\rangle_{\mathrm{a}} |n\rangle_{\mathrm{f}} = |A - N + n\rangle_1 |N - n\rangle_2 |n\rangle_{\mathrm{f}} \tag{6.15}$$

$$0 \le n \le N, \quad N < A$$

participate in the dynamics; the dimension of this subspace is $N + 1$. When $A > N$, there are many photons in the "idler" mode, $|k\rangle_1$. In the subspace (6.15), the operators b_1 and $b_1^\dagger$ have the matrix elements $\sqrt{k}$, where $A - N + 1 \leq k \leq A$. We may replace these operators by a fictitious classical field with the intensity equal to the "average" photon number in the above interval:

$$\sqrt{A - \frac{N}{2} + \frac{1}{2}} = \Omega_N/g, \quad N < A \tag{6.16}$$

Ω_N plays the role of the Rabi frequency. The zeroth-order approximation for the interaction Hamiltonian may be written in the form

$$V_0 = \Omega_N(a b_2^\dagger + a^\dagger b_2) \tag{6.17}$$

Another justification for this choice of the zeroth-order approximation is given below. In the subspace (6.15), the operators

$$L_+ = a b_2^\dagger = L_-^\dagger, \quad L_z = \frac{(b_2^\dagger b_2 - a^\dagger a)}{2} \tag{6.18}$$
$$L_x = \frac{(L_+ + L_-)}{2}, \quad L_y = \frac{(L_+ - L_-)}{2i}$$

determine the $(N+1)$-dimensional representation of a new "dynamic" algebra $su(2)$. The zeroth-order Hamiltonian (6.17) can be also written as

$$V_0 = 2\Omega_N L_x$$

In other words, the zeroth-order dynamics is connected with the rotation of the fictitious spin $N/2$ around the x axis with the frequency $2\Omega_N$. Now, in every subspace with a given N, we may construct a perturbation theory, treating the operator $V - V_0$ as a perturbation.

6.4.2
The Strong Field Case

If the photon number is larger than the number of atoms, $A < N$, then the states

$$|N, n\rangle = |A,\ N - n\rangle_{\mathrm{a}} |n\rangle_{\mathrm{f}}, \quad N - A \leq n \leq N \tag{6.19}$$

are involved in the evolution, and the dimension of the subspace with N excitations is $A + 1$. If $A \ll N$, the number of real photons is large and the zeroth-order approximation corresponds to a classical field with the intensity equal to the "average" number of photons in the interval $[N - A + 1,\ N]$:

$$\sqrt{N - \frac{A}{2} + \frac{1}{2}} = \Omega_N/g, \quad A < N \tag{6.20}$$

which gives us the carrying Rabi frequency in this case. Note, that the strength of the effective classic field depends on the index of the subspace N. The dynamic algebra operators $L_{\pm,z}$, in this case, coincide with the atomic operators (1.12). The frequencies given by Equations 6.16 and 6.20 coincide when $A = N$.

It is easy to find the eigenvectors and eigenvalues in the zeroth-order approximation:

$$V_0|\underline{N,p}\rangle = \Lambda^{(0)}_{Np}|\underline{N,p}\rangle$$

The zeroth-order Hamiltonian V_0 in the weak (strong) field limit is proportional to $2L_x$ from the $(N+1)$-dimensional (correspondingly, $(A+1)$-dimensional) representation of the $su(2)$ algebra. It can be diagonalized by the rotation around the y axis through an angle $\pi/2$:

$$\exp\left(\mathrm{i}\frac{\pi}{2}L_y\right) L_x \exp\left(-\mathrm{i}\frac{\pi}{2}L_y\right) = L_z$$

The operators L_x and L_z have the same equidistant spectrum, $N/2 - p$, $p = 0, 1, \ldots, N$. (Here we write the formulae for the weak field case; those for the strong field case can be obtained by interchanging N and A.) The zeroth-order eigenvalues of the Hamiltonian are

$$\Lambda^{(0)}_{Np} = \Omega_N(N - 2p), \quad p = 0, 1, \ldots, N$$

where the collective Rabi frequency is defined as follows:

$$\Omega_N = \begin{cases} g\sqrt{N - \dfrac{A}{2} + \dfrac{1}{2}}, & A \le N \\ g\sqrt{A - \dfrac{N}{2} + \dfrac{1}{2}}, & A \ge N \end{cases} \tag{6.21}$$

The eigenvectors in the zeroth-order approximation are

$$|\underline{N,p}\rangle = \exp\left(\mathrm{i}\frac{\pi}{2}L_y\right)|N,p\rangle \tag{6.22}$$

and the components in the bare basis are given by the Wigner d functions of argument $\pi/2$ (e.g. [70]):

$$\alpha^N_{np} = \langle N,n|\underline{N,p}\rangle = \left(\frac{n!p!}{2^N(N-n)!(N-p)!}\right)^{1/2} \times \sum_{j=0}^{\min(p,n)} \frac{(-2)^j(N-j)!}{j!(n-j)!(p-j)!} \tag{6.23}$$

Note that α^N_{np} do not depend on the atomic number A. Recall that this formula is written for the weak field case. In the strong field case, it is necessary to replace N by A.

6.5
Perturbation Theory

We describe here how to find corrections to the equidistant spectrum of the zeroth-order approximation [71–73]. The interaction Hamiltonian (6.13) in the basis (6.15) or (6.19) is reduced to the three-diagonal matrix with nonvanishing matrix elements

$$\begin{aligned}\langle N, n-1|V|N, n\rangle &= \langle N, n|V|N, n-1\rangle \\ &= \sqrt{n(N-n+1)(A-N+n)}\end{aligned} \tag{6.24}$$

When $A \gg N$ and $0 \leq n \leq N$, the factor $A - N + n$ is large and the square root can be expanded in series with the small parameter

$$\epsilon = \frac{1}{A - N/2 + 1/2} = \frac{g^2}{\Omega_N^2}, \quad A > N \tag{6.25}$$

In turn, in the strong field case, $N \gg A$ and $N - A \leq n \leq N$, the factor n is large and one may expand the square root using the small parameter

$$\epsilon = \frac{1}{N - A/2 + 1/2} = \frac{g^2}{\Omega_N^2}, \quad N > A \tag{6.26}$$

The interaction Hamiltonian is then written as

$$V = 2\Omega_N(V_0 + \epsilon V_1 + \epsilon^2 V_2 + \cdots) \tag{6.27}$$

In the *Weak field case,* $N \leq A$, the nonvanishing matrix elements of the operators in Equation 6.27 are

$$\begin{aligned}\langle N,\ n|V_0|N,\ n-1\rangle &= \sqrt{n(N-n+1)} \\ \langle N,\ n|V_1|N,\ n-1\rangle &= \frac{1}{2}\left(n - \frac{N}{2} - \frac{1}{2}\right)\sqrt{n(N-n+1)} \\ \langle N,\ n|V_2|N,\ n-1\rangle &= -\frac{1}{8}\left(n - \frac{N}{2} - \frac{1}{2}\right)^2\sqrt{n(N-n+1)}\end{aligned}$$

This expansion can be rewritten in the operator form. To do so, we consider the $(N+1)$-dimensional representation of the dynamic $su(2)$ algebra whose generators $L_{\pm,z}$ (compare with Equation 6.18) have the following form in the basis $|N,\ n\rangle$:

$$L_+|N,\ n\rangle = \sqrt{(n+1)(N-n)}|N,\ n+1\rangle \tag{6.28}$$

$$L_-|N,\ n\rangle = \sqrt{n(N-n+1)}|N,\ n-1\rangle \tag{6.29}$$

$$L_z|N,\ n\rangle = \left(n - \frac{N}{2}\right)|N,\ n\rangle \tag{6.30}$$

Taking into account the matrix elements in Equation 6.24, the interaction Hamiltonian (6.13) is written in terms of the operators (6.28) to (6.30) as

$$\begin{aligned}V &= g\left[a^\dagger S_- + aS_+\right] \\ a^\dagger S_- &= \epsilon^{-1/2} L_+ \sqrt{1 + \epsilon\left(L_z + \frac{1}{2}\right)} \\ a^\dagger S_- &= \epsilon^{-1/2}\sqrt{1 + \epsilon\left(L_z + \frac{1}{2}\right)}L_-\end{aligned}$$

Expanding the square roots in Taylor series and taking into account that

$$L_+\left(L_z+\frac{1}{2}\right)+\left(L_z+\frac{1}{2}\right)L_- = \{L_z, \quad L_x\}$$

$$L_+\left(L_z+\frac{1}{2}\right)^2+\left(L_z+\frac{1}{2}\right)^2 L_- = \frac{1}{2}\{L_z, \quad \{L_z, L_x\}\}$$

we can rewrite Equation 6.27 as

$$V = 2\Omega_N\left[L_x+\frac{\epsilon}{4}\{L_z, L_x\}-\frac{\epsilon^2}{32}\{L_z,\ \{L_z,\ L_x\}\}+O(\epsilon^3)\right] \tag{6.31}$$

which corresponds to the perturbative form (6.27).

We use $\Lambda_{N,p}$ and $|\Psi_{N,p}\rangle$ to denote the eigenvalues and eigenvectors of the interaction Hamiltonian

$$V|\Psi_{N,p}\rangle = \Lambda_{N,p}|\Psi_{N,p}\rangle$$

and A_{np} to denote the matrix elements of the transition from the basis $|N, n\rangle$ to the basis $|\Psi_{N,p}\rangle$, $A_{np} = \langle N, n|\Psi_{N,p}\rangle$. The Hamiltonian in the zeroth-order approximation is proportional to L_x. The eigenvalues and eigenvectors in the zeroth-order approximation are given by Equation 6.22.

It is easy to see that the first-order corrections to the eigenvalues vanish, due to the form of the zeroth-order approximation:

$$\langle \underline{N,p}|V_1|\underline{N,p}\rangle = \langle \underline{N,p}|\{L_z,\ L_x\}|\underline{N,p}\rangle = 0$$

This is a consequence of our choice of Ω_N in Equation 6.17 and of the small parameter Equation 6.25. In fact, this cancellation of the first-order corrections is the reason of our choice of small parameters (6.25) and (6.26).

Using the standard perturbation theory, we find eigenvalues in the second order

$$\Lambda^{(2)}_{N,p} = \Omega_N(N-2p)\left\{1-\frac{\epsilon^2}{32}\left[10p(N-p)-(N-1)(N-2)\right]\right\}$$

where $0 \le p \le N$; Ω_N and ϵ have the form (6.21) and (6.25), respectively.

For the cases $N = 1$ and $N = 2$, the second-order corrections (and corrections of all higher orders) vanish and the spectrum is equidistant for any A. For $N \ge 3$ the spectrum is not equidistant due to the nonlinear dependence on the index p. The components of the first-order eigenvector are given by

$$\begin{aligned} A^{(1)}_{np} = \alpha^N_{np} + \frac{\epsilon}{8}\Big\{&(N-2p+1)\sqrt{p(N-p+1)}\alpha^N_{np-1} \\ &-(N-2p-1)\sqrt{(p+1)(N-p)}\alpha^N_{np+1}\Big\} \end{aligned} \tag{6.32}$$

where the zeroth-order components α^N_{np} are defined by Equation 6.23.

In the *strong field case*, $A \leq N$. It is convenient to change the notation for the basis (6.19) as

$$\begin{aligned}&\underline{n} = n - N + A\\ &|N,\ n\rangle \to |N,\ \underline{n}\rangle = |A - \underline{n}\rangle_{\mathrm{a}} |\underline{n} + N - A\rangle_{\mathrm{f}}\end{aligned} \tag{6.33}$$

where $\underline{n}$ denotes the number of atoms in the ground state and varies in the interval $0 \leq \underline{n} \leq A$. Thus, the matrix elements of the interaction Hamiltonian for the case of the strong field in the basis (6.33) are

$$\langle N,\ \underline{n} - 1|V|N,\ \underline{n}\rangle = \sqrt{(\underline{n} + N - A)\,(A - \underline{n} + 1)\,\underline{n}}$$

This equation can be obtained directly from (6.24) on replacing n by $\underline{n}$ and interchanging A and N. The appropriate formulae for the case of the strong field (we use the basis (6.33) instead of (6.19)) can be obtained from the formulae for the weak field, exchanging A and N. The operators of the dynamic algebra in the case of the strong field coincide with the atomic operators (1.12), $S_{\pm,z}$. Thus in Equation 6.31, we have to replace $L_{\pm,z}$ by $S_{\pm,z}$. Once again, the first-order corrections to the eigenvalues vanish:

$$\underline{\langle N,\ p|}V_1\underline{|p,\ N\rangle} = 0 \tag{6.34}$$

and, in the second order, we have

$$\begin{aligned}\Lambda_{N,p}^{(2)} &= 2\Omega_N \left(\frac{A}{2} - p\right)\left\{1 - \frac{\epsilon^2}{32}\left[10p(A-p) - (A-1)(A-2)\right]\right\}\\ &\quad 0 \leq p \leq A\end{aligned} \tag{6.35}$$

The nonlinear frequencies given by Equations 6.32 and 6.35 are called the *Dicke frequencies* to highlight the difference between these and the Rabi frequencies Ω_N. The components of the eigenvectors in the first-order approximation are

$$\begin{aligned}A_{\underline{n}p}^{(1)} = \alpha_{\underline{n}p}^{A} + \frac{\epsilon}{8}\Big\{&(A - 2p + 1)\sqrt{p(A - p + 1)}\alpha_{\underline{n}p-1}^{A}\\ &-(A - 2p - 1)\sqrt{(p + 1)(A - p)}\alpha_{\underline{n}p+1}^{A}\Big\}\end{aligned} \tag{6.36}$$

where the components of the zeroth-order eigenvectors $\alpha_{\underline{n}p}^{A}$ are given by the formula (6.23) replacing n by $\underline{n}$ and N by A.

As already mentioned, the zeroth-order approximation reproduces the exact solution for the JCM ($A = 1$) and yields the exact spectrum for $A = 2$; on the other hand, the zeroth-order eigenvectors for $A \geq 2$ differ from the exact ones in the terms of order $1/N$.

6.6
Revivals of the First and Second Orders

We first make some general remarks about revivals that are not restricted to the Dicke model. Let the evolution of some observable be given by an anharmonic series with frequencies of oscillations $2\Omega_n$. (Here, we include the factor 2 to retain the correspondence with the JCM notations.) Relative contributions ("the distribution") of these frequencies are determined by the initial conditions. Assume that this distribution has its maximum at the point $n = \overline{n}$ and the spread is Δn. If the frequency Ω_n is a smooth function of n, we may expand it around the point $\overline{n}$:

$$\Omega_n = \Omega_{\overline{n}} + (n - \overline{n})\Omega_{\overline{n}}^{(1)} + (n - \overline{n})^2\Omega_{\overline{n}}^{(2)} + \cdots$$
$$\Omega_{\overline{n}}^{(r)} = \frac{1}{r!}\frac{\mathrm{d}^r\Omega_n}{\mathrm{d}n^r}\bigg|_{n=\overline{n}} \tag{6.37}$$

The first term here leads to rapid oscillations, while the remaining terms determine their envelope. If $|\Omega_{\overline{n}}^{(1)}| \gg |\Omega_{\overline{n}}^{(k)}|$, $k = 2, 3, \ldots$, then the linear term is responsible for collapses and revivals of the oscillations. We can find the revival time by the same arguments as for the JCM with a coherent field. At the revival time, the most heavily weighted oscillations are in phase:

$$2|\Omega_{\overline{n}+1} - \Omega_{\overline{n}}|T_{\mathrm{R}} \approx 2|\Omega_{\overline{n}}^{1}|T_{\mathrm{R}} = 2\pi \tag{6.38}$$

It is easy to see that the terms with frequencies $2\Omega_{\overline{n}+k}$ and $2\Omega_{\overline{n}}$ are also in phase at that time instant, provided that $k \ll \Delta n$. This leads to an amplification of the oscillations. Therefore, the revival time is determined by the first derivative of the frequency Ω_n (which plays the role of a dispersion curve) at the point $\overline{n}$. For this reason, we refer to this process as revivals of the first order [74]. JCM revivals are of the first order, as it follows from Chapter 5. In the linear (harmonic) approximation revivals are perfectly periodic. The higher-order terms in Equation 6.37 lead to the growing spread and decreasing amplitudes of subsequent revivals. For some special initial distributions, the higher-order terms can significantly modify the shape of revivals, as for ringing revivals in the JCM with the initial field in a strongly squeezed state [75].

We now apply these arguments to the dynamics of the Dicke model in the weak field limit, $A \gg N$, so that we may restrict ourselves to the zeroth-order approximation. Assume that all the atoms are initially in their ground states. Then the photon number distribution $P(n)$ in the initial state is transformed to the excitation-number distribution $P(N)$, which determines the weights of different Rabi frequencies that contribute to the evolution. We show the zeroth-order Rabi frequencies Ω_n (6.21) in Figure 6.1 for the case $A = 100$, $0 \leq n \leq 200$, which covers different dynamic regions. Interference of different frequencies $2\Omega_n$ may lead to collapses and revivals of oscillations in this system. We now find the conditions when this happens. We can rewrite the linear terms in the expansion (6.37) for the Rabi frequency (6.21) in a slightly different form:

$$\Omega_N \approx \Omega_{\overline{n}} + (n - \overline{n})\Omega_{\overline{n}}^{(1)} \approx \Omega_0 - n\Omega_{\overline{n}}^{(1)} \tag{6.39}$$

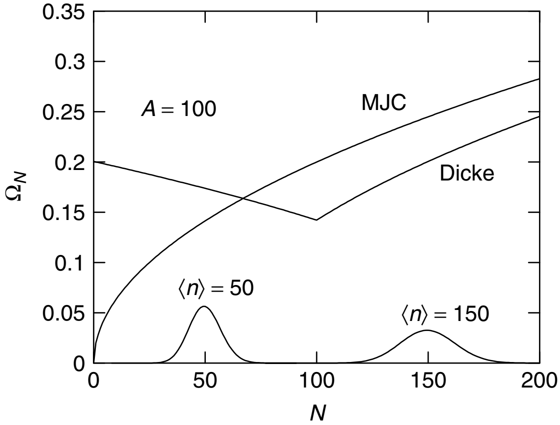

Figure 6.1 Rabi frequency, $2\Omega_N$ as a function of the excitation number, N for $A = 100$ atoms ($g = 0.1$); Rabi frequency for the JCM ($A = 1$); Poisson distributions for $\langle n \rangle = 50$ and $\langle n \rangle = 150$.

where now,

$$\Omega_0 = \Omega_{n=0} = g\sqrt{A + \frac{1}{2}}$$

while

$$\Omega^{(1)}_{\overline{n}} = \left|\frac{\mathrm{d}\Omega}{\mathrm{d}n}\right|_{n=\overline{n}} = \frac{g}{4\sqrt{A - \dfrac{\overline{n}}{2} + \dfrac{1}{2}}}$$

From Equation 6.38, it follows that the revival time depends only on the number of atoms

$$T^{(w)}_{\mathrm{R}} = \frac{\pi}{\Omega^{(1)}_{\overline{n}}} \approx \frac{4\pi\sqrt{A}}{g}$$

We show the Poissonian distribution with $\overline{n} = 50$ in Figure 6.1 as an example of the weak field limit. Note, that in this limit of the Dicke model, $\overline{n} \ll A$, collapses and revivals also appear for an initial thermal state of the field (in contrast to the JCM see Section 5.6).

In the strong field limit, $\overline{n} \gg A$, from Equations 6.21 and 6.38, we obtain the revival time

$$T^{(s)}_{\mathrm{R}} = \frac{\pi}{\Omega^{(1)}_{\overline{n}}} \approx \frac{2\pi\sqrt{\overline{n}}}{g}$$

The Poissonian distribution with $\overline{n} = 150$ is shown in Figure 6.1 as an example of the strong field regime.

For a single atom case, $A = 1$, we obtain from Equation 6.21, the JCM Rabi frequency, $2\Omega_n = 2g\sqrt{n}$, which is also shown in Figure 6.1. The JCM is a particular case of the Dicke model in the strong field regime. It is well known that for the on-resonant JCM with an initially coherent field, collapses and revivals disappear for small photon numbers. The reason is that in this case, the sum over the photon numbers includes the region of the greatest nonlinearity of the Rabi frequency. In fact, the derivative of Ω_n at the point $n = 0$ is singular and the linear approximation fails. This is also the reason for the absence of collapses and revivals in the JCM with the initial thermal field, [50] and squeezed vacuum field [76] for any value of $\overline{n}$. In turn, collapses and revivals occur in the Dicke model with initial thermal field if it belongs to the weak field region [77].

It is important to stress that the formula (6.38) is valid for various field states to which both odd and even photon numbers contribute. For an initial squeezed vacuum field (to which only even photon number states contribute), the revival time in the Dicke model is half the time in Equation 6.38 [78]. In this case, the nearest-neighbor term to $\overline{n}$ is not $\overline{n}+1$ but $\overline{n}+2$ and Equation 6.38 must be modified, correspondingly.

The behavior is completely different if the first derivative of the frequency Ω_n vanishes: $\Omega_1^{(\overline{n})} = 0$. In general, one can speak of revivals of the kth order if all of the derivatives $\Omega_r^{(\overline{n})}$, $r = 1, 2, \ldots, k-1$ are zeros, and the first nonvanishing derivative is $\Omega_k^{(\overline{n})}$. In particular, if $\Omega_2^{(\overline{n})} \neq 0$ (and $\Omega_1^{(\overline{n})} = 0$), we are faced with revivals of the second order. The shape of these revivals is different and the formula (6.38) is no longer valid.

6.6.1
Revivals of the Second Order

As an example of second-order revivals, we consider the case when the field is taken initially in a Fock state $|n\rangle_f$, and the atomic system is prepared in its ground state, $|A, 0\rangle$. We consider the weak field case, $n \ll A$. Thus, the initial state of the total system is $|in\rangle = |n,\ n\rangle$ and $N = n$. The evolution of the average photon number is given in terms of eigenvalues (6.32) and eigenvectors (6.32) as follows:

$$\langle \hat{n}(t) \rangle = \sum_{p,\ q=0}^{n} \left(\sum_{m=0}^{n} m A_{mp} A_{mq} \right) A_{np} A_{nq} \mathrm{e}^{\mathrm{i}(\Lambda_{n,\ q} - \Lambda_{n,p})t}$$

We keep the second-order accuracy in eigenvalues, restricting ourselves to the first-order accuracy in eigenvectors (as indicated by a subindex 21). From Equations 6.32 and 6.32, we have

$$\begin{aligned} \langle n(t) \rangle_{21} = {} & \frac{n}{2} - \frac{\epsilon n(n-1)}{16} + \sum_{p=1}^{n} C_{np} \cos 2\Omega^{(p)} t \\ & + \frac{\epsilon}{8} \sum_{p=1}^{n} C_{np} \left\{ \left[(n-2p)^2 - 2p + 1 \right] \cos 2\Omega^{(p)} t \right. \\ & \left. + 2(p-1) \cos 2\tilde{\Omega}^{(p)} t \right\} \end{aligned} \qquad (6.40)$$

Here the principal term (of the zeroth-order in eigenvectors) contains the second-order Dicke frequencies

$$\begin{aligned} 2\Omega^{(p)} &= \Lambda^{(2)}_{n,\,p-1} - \Lambda^{(2)}_{n,\,p} \\ &= 2\Omega_n \left\{ 1 + \frac{3\epsilon^2}{16}[5(p-1)(p-n) + (n-1)(n-2)] \right\} \end{aligned} \tag{6.41}$$

weighted with the coefficients

$$C_{np} = \frac{n!}{2^n (n-1)!(p-1)!}$$

which are the binomial distribution multiplied by the factor $n/2^n$. In the first-order approximation in eigenvectors, new frequencies appear

$$2\tilde{\Omega}^{(p)} = \Lambda^{(2)}_{n,p-2} - \Lambda^{(2)}_{n,p} \tag{6.42}$$

The oscillation amplitudes depend on the atomic number only through the small parameter ϵ.

It is clear from Equation 6.40–6.42 that the most important role in the evolution is played by the terms with $p \sim n/2$. The Dicke frequency (6.41), considered as a continuous function of p, takes its maximum value at the point $p_m = (n+1)/2$. The distribution C_{np} takes its maximum value at the same point (see Figure 6.2).

Therefore, revivals, which occur (see Figure 6.3), are of the second order. The shape of these revivals is very different from that of the first order. The discussion of their properties can be found in [74, 79]. (These phenomena were termed "superstructures" in [79].) We mention here that the revival time depends on the parity of the initial photon number n. Namely, the revival time for the same atomic

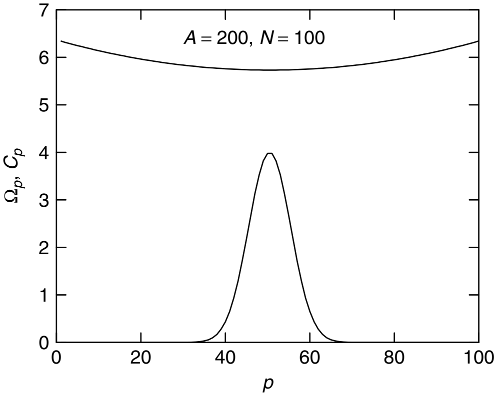

Figure 6.2 Collective Dicke frequency, $2\Omega^{(p)}$, Equation 6.26, for $A = 200$, $N = 100$ and $g = 10^{-2}$ and the binomial distribution of frequencies.

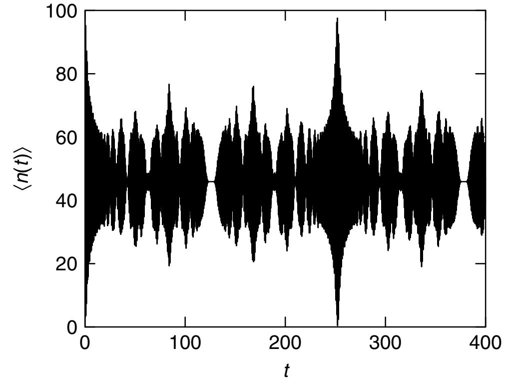

Figure 6.3 Revivals of the second order in the Dicke model: photon number evolution for the initial Fock field state with 100 photons and 200 atoms initially taken in their ground states.

number A and odd n is twice the revival time for the neighboring even n:

$$T_{R\ \mathrm{odd}\ n} = \frac{16\pi}{15g}\left(A - \frac{n}{2} + \frac{1}{2}\right)^{3/2} \tag{6.43}$$

$$T_{R\ \mathrm{even}\ n} = \frac{8\pi}{15}\left(A - \frac{n}{2} + \frac{1}{2}\right)^{3/2} \tag{6.44}$$

Collapses and revivals occur periodically in the approximation discussed.

The second-order revivals appear also in the JCM with a Kerr medium for special values of the parameters [79]. Then initially coherent field states lead to different revival times depending on whether the mean photon numbers in the initial coherent state is an integer or a fraction.

We note that the Rabi frequency is not a smooth function at the point $N = A$, as it is seen in Figure 6.1. Thus, the vicinity of this point is beyond the scope of the perturbation method under discussion. In this case, it is necessary to use other approaches (see, for instance, [80–82]).

6.7
Atom-Field Dynamics for Different Initial Conditions

6.7.1
Initial Number States

For the same case as in the end of the previous section, when, initially, all the atoms are in their ground states and the field is in the number state with n photons, we discuss the *photon statistics*. We start with the *weak field regime*. The expected value of the square of the operator of the number of photons, in the second-order approximation for the eigenvalues and in the zeroth-order approximation for the eigenvectors (subindex 20), is

$$\langle n^2(t)\rangle_{20} = \frac{n(3n+1)}{8} + \sum_{p=1}^{n} C_p^n \left\{ n\cos\left[(\Lambda^{(2)}_{n,p-1} - \Lambda^{(2)}_{n,p})t\right] + \frac{p-1}{2} \times \cos\left[\left(\Lambda^{(2)}_{n,p-2} - \Lambda^{(2)}_{n,p}\right)t\right]\right\} \tag{6.45}$$

For a sufficiently small n compared to A, the zeroth-order approximation (subindex 00) is sufficient:

$$\langle n(t)\rangle_{00} = \frac{n}{2}(1 + \cos\Omega_n t) \tag{6.46}$$

$$\langle n^2(t)\rangle_{00} = \frac{n}{8}\left[1 + 3n + 4n\cos\Omega_n t + (n-1)\cos 2\Omega_n t\right]$$

For $n = 1$, the formula above yields the exact solution.

In terms of the second-order correlation function $G^{(2)}(t) = \langle n^2(t)\rangle - \langle n(t)\rangle$, the second-order coherence function $g^{(2)}$ is of the form

$$g^{(2)}(t) = \frac{G^{(2)}(t)}{\langle n(t)\rangle^2} = 1 + \frac{\mathcal{V}(t)}{\langle n(t)\rangle^2}$$

where $\mathcal{V}$ is the variance of the normally ordered number of photons, $\mathcal{V}(t) = G^{(2)}(t) - \langle n(t)\rangle^2$. The negative values of the variance $\mathcal{V}$, in this case $g^{(2)} < 1$, indicates the sub-Poissonian distribution of the field. Fields of this type have no classical analog.

Note that at least two photons are necessary for the correlation function $G^{(2)}(t)$ to be different from zero. For $n = 2$, the variance $\mathcal{V}$ cannot be positive and the field is sub-Poissonian. However, for $n = 3$, the statistics for the number of photons oscillates between the sub-Poissonian and super-Poissonian.

We now consider the *strong field regime*. From here on, we skip the underlining $\underline{n}$ in the sums for the strong field case. The formulae for the average number of photons in the case of the strong initial field in the number state are obtained basically from the case of the weak field, by interchanging n and A. We obtain,

$$\langle n(t)\rangle_{21} = n - \frac{A}{2} - \frac{\epsilon A(A-1)}{16} + \sum_{p=1}^{A} C_p^A \left\{ \left(1 + \frac{\epsilon}{8}[(A-2p)^2 - 2p + 1]\right)\cos\left[\left(\Lambda^{(2)}_{n,p} - \Lambda^{(2)}_{n,p-1}\right)t\right] + \frac{\epsilon(p-1)}{4}\cos\left[\left(\Lambda^{(2)}_{n,p} - \Lambda^{(2)}_{n,p-2}\right)t\right]\right\}$$

The frequencies $\Lambda_{n,p}^{(2)}$ are defined by Equation 6.35. In this regime, the oscillation amplitudes of the average number of photons depend only on n, through the small parameter ϵ. In general, the time behavior of $\langle n(t)\rangle_{21}$ is similar to the case of the weak initial field in the number state. In particular, the second-order revivals take place (like in the case of the weak field).

Note that the revival time of the principal term $\cos[(\Lambda_{n,p}^{(2)} - \Lambda_{n,p-1}^{(2)})t]$ is also related with the parity, but now of the number of atoms A. The revival time for the odd number of atoms is

$$T_{\mathrm{R\ odd}\ A} = \frac{16\pi}{15g}\left(n - \frac{A}{2} + \frac{1}{2}\right)^{3/2}$$

while for the even number of atoms, it is

$$T_{\mathrm{R\ pair}\ A} = \frac{8\pi}{15g}\left(n - \frac{A}{2} + \frac{1}{2}\right)^{3/2}$$

The collective effects appear for the numbers of atoms $A \geq 3$.

As in the case of the weak field, one can find the normalized variance of the number of photons in the field. In the zeroth-order approximation we have

$$\mathcal{V}_{00}(t) = -n + \frac{A}{8}\left(5 - 4\cos\Omega_n t - \cos 2\Omega_n t\right)$$

which implies that the field remains sub-Poissonian always.

6.7.2
Coherent and Thermal Fields

To calculate the average number of photons for the cases when the field is initially prepared in the coherent or thermal states, we have to sum up the mean values obtained for the number state with the initial photon distribution P_n:

$$\langle n(t)\rangle = \sum \langle n(t)\rangle_n P_n$$

Here $\langle n(t)\rangle_n$ is the average over the number state, and the photon number distributions are given by

$$P_n = \exp(-\overline{n})\frac{\overline{n}^n}{n!} \quad \text{or} \quad P_n = \frac{\overline{n}^n}{(1+\overline{n})^{n+1}}$$

In Figure 6.4a,b the oscillations are drawn for the cases of the initial state of the thermal and coherent fields with the average number of photons $\overline{n} = 4$ and the number of atoms $A = 100$ (regime of the weak field).

In the zeroth-order approximation we have

$$\langle n(t)\rangle_{00}^{(d)} = \frac{1}{2}\left(\overline{n} + \sum_{n=1}^{n\ll A} nP_n \cos 2\Omega_n t\right) \tag{6.47}$$

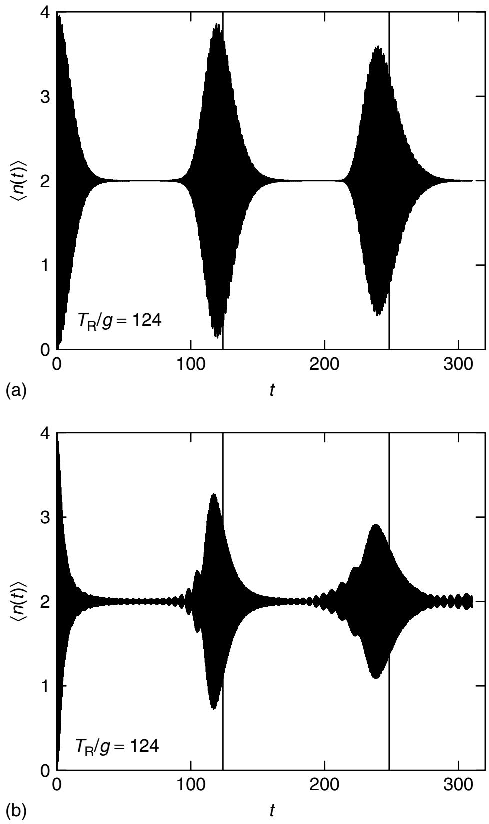

Figure 6.4 (a) Photon number oscillations for initial coherent state of the field with $\bar{n} = 4$ and $A = 100$. (b) Photon number oscillations for initial thermal state of the field with $\bar{n} = 4$ and $A = 100$.

where the Rabi frequencies Ω_n are defined in (6.21). This formula does not include the collective effects and is valid for times of order $gt\Omega\epsilon^2 \sim gtA^{-3/2} < 1$. For longer times, the collective effects appear due to the second-order corrections in the Dicke frequencies given by Equation 6.32. In the *weak field limit*, when the average number of photons in the field is very small in comparison to the number of atoms, then, for fields in the thermal and coherent state, a very regular structure of collapses and revivals can be observed. To find an analytic approximation for Equation 6.47 for the case $A \gg \bar{n}$, we linearize the Rabi frequency (6.39). For the initial coherent state of the field, we obtain for the evolution of the mean number of photons

$$\langle n(t)\rangle_{\mathrm{coh}}^{(d)} \approx \frac{\bar{n}}{2}\left(1 + \mathrm{e}^{-\bar{n}(1-\cos\tau)}\cos(T - \tau - \bar{n}\sin\tau)\right)$$

where $T \equiv 2\Omega_0 t$ and $\tau \equiv 2\Omega_{\mathrm{R}} t$ naturally determine two time scales. Here, $\Omega_0 = g\sqrt{A + 1/2}$, $\Omega_{\mathrm{R}} = g/4\sqrt{A - \bar{n}/2 + 1/2}$. The envelope of the oscillations is of the form

$$\langle n(t)\rangle^{(d)}_{\text{env coh}} = \frac{\overline{n}}{2}\left\{1 \pm \exp[-\overline{n}(1-\cos\tau)]\right\}$$

It revives periodically at times $gt = kT^{(d)}_{\text{R}}$, $(k = 1, 2, \ldots)$, where $T_{\text{R}} = \pi/\Omega_{\text{R}}$. For the initial thermal field, we obtain a different expression

$$\langle n(t)\rangle^{(d)}_{\text{env tf}} = \frac{\overline{n}}{2}\left\{1 \pm [1 + 2\overline{n}(\overline{n}+1)(1-\cos\tau)]^{-1}\right\}$$

In the *strong field regime*, the evolution of the average number of photons in the zeroth-order approximation is given as follows:

$$\langle n(t)\rangle^{(f)}_{00} = \overline{n} - \frac{A}{2}\left(1 - \sum_{n \gg A}^{\infty} P_n \cos 2\Omega_n t\right) \tag{6.48}$$

Making the linear approximation in the frequencies given by Equation 6.37, we obtain from Equation 6.48, for a coherent initial state of the field

$$\langle n(t)\rangle^{(f)}_{\text{cf}} = \overline{n} - \frac{A}{2}\left[1 - \mathrm{e}^{-\overline{n}(1-\cos\tau)}\cos(T - \overline{n}\tau + \overline{n}\sin\tau)\right]$$

where $T = 2\Omega_{\overline{n}}t$, $\tau = g^2 t/\Omega_{\overline{n}}$ and the frequency $\Omega_{\overline{n}} = g(\overline{n} - A/2 + 1/2)^{1/2}$.

6.8
Three-Level Atoms Interacting with Two Quantum Field Modes

The perturbation theory developed in this chapter can be generalized to more complicated atomic systems interacting with quantum fields. As an example, we consider a collection of identical three-level atoms in the V-configuration (with allowed transitions $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$) interacting with two modes of the quantum field. We restrict ourselves to the strong field case (when number of photons in the field modes are much greater than the number of atoms, $n_1, n_2 \gg A$) under the exact resonance condition, when the atomic transition frequencies are exactly equal to the field frequencies: $\omega_1 = E_2 - E_1$, and $\omega_2 = E_3 - E_1$ $(\hbar = 1)$.

The Hamiltonian governing the system dynamics has the form

$$H = \omega_1\left(a_1^{\dagger}a_1 + S_z^{12}\right) + \omega_2\left(a_2^{\dagger}a_2 + S_z^{13}\right) + H_{\text{int}} \tag{6.49}$$

$$H_{\text{int}} = g_1\left(S_+^{12}a_1 + S_-^{12}a_1^{\dagger}\right) + g_2\left(S_+^{13}a_2 + S_-^{13}a_2^{\dagger}\right) \tag{6.50}$$

where we choose the coupling constants $g_{1,2}$ to be real, for simplicity. Here, $S^{12}_{\pm,z}$, $S^{13}_{\pm,z}$ are the collective operators (1.48) describing the collection of A three-level atoms (see Section 1.5.2). They obey the commutation relations of the $su(3)$ algebra. The $su(3)$ algebra has eight generators and, correspondingly, 28 commutators. The operators S_z^{12}, S_z^{13} form a commutative subalgebra (the so-called Cartan subalgebra), $\left[S_z^{12}, S_z^{13}\right] = 0$. The operators $S^{12}_{\pm,z}$ and $S^{13}_{\pm,z}$ form two $su(2)$ subalgebras

$$\begin{aligned} \left[S_z^{12},\ S_\pm^{12}\right] &= \pm S_\pm^{12}, \quad \left[S_+^{12},\ S_-^{12}\right] = 2S_z^{12} \\ \left[S_z^{13},\ S_\pm^{13}\right] &= \pm S_\pm^{13}, \quad \left[S_+^{13}, S_-^{13}\right] = 2S_z^{13} \end{aligned} \tag{6.51}$$

and

$$\left[S_z^{12},\ S_\pm^{13}\right] = \pm\frac{1}{2}S_\pm^{13},\quad \left[S_z^{13},\ S_\pm^{12}\right] = \pm\frac{1}{2}S_\pm^{12} \tag{6.52}$$

Two more generators are required to write the commutation relations in the closed form:

$$\left[S_+^{12}, S_-^{13}\right] = S_-^{23},\quad \left[S_+^{13},\ S_-^{12}\right] = S_+^{23} \tag{6.53}$$

The rest of commutation relations are

$$\begin{aligned}
&\left[S_z^{12},\ S_\pm^{23}\right] = \mp\frac{1}{2}S_\pm^{23},\quad \left[S_z^{13},\ S_\pm^{23}\right] = \pm\frac{1}{2}S_\pm^{23},\\
&\qquad\qquad \times\left[S_+^{23},\ S_-^{23}\right] = 2\left(S_z^{13} - S_z^{12}\right)\\
&\left[S_+^{12},\ S_+^{23}\right] = -S_+^{13},\quad \left[S_-^{12},\ S_-^{23}\right] = S_-^{13},\quad \left[S_+^{13},\ S_-^{23}\right] = -S_+^{12},\\
&\qquad\qquad \times\left[S_-^{13},\ S_+^{23}\right] = S_-^{12}\\
&\left[S_+^{12},\ S_-^{23}\right] = \left[S_+^{12},\ S_+^{13}\right] = \left[S_-^{12},\ S_-^{13}\right] = \left[S_-^{12},\ S_+^{23}\right] = \left[S_+^{23},\right.\\
&\qquad\qquad \left.\times\ S_+^{13}\right] = \left[S_-^{23},\ S_-^{13}\right] = 0
\end{aligned}$$

The case of a single atom corresponds to the three-dimensional representation of the *su*(3) algebra, given by

$$\begin{gathered}
S_z^{12} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1 \end{bmatrix},\quad S_+^{12} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0 \end{bmatrix},\quad S_-^{12} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0 \end{bmatrix}\\
S_z^{13} = \begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1 \end{bmatrix},\quad S_+^{13} = \begin{bmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix},\quad S_-^{13} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0 \end{bmatrix}\\
S_+^{23} = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix},\quad S_-^{23} = \begin{bmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}
\end{gathered} \tag{6.54}$$

For a system of A identical three-level atoms, when only the symmetrical states with respect to permutations of atoms are considered, the collective atomic operators form the $(A+1)(A+2)/2$-dimensional irreducible representation of *su*(3). The matrix elements of the generators in this representation can be found, for instance, by using Schwinger realization of the *su*(3)(Equation 1.50),

$$\begin{aligned}
S_+^{12}\,|m_1,\,m_2,\,m_3\rangle &= \sqrt{m_1(m_2+1)}\,|m_1-1,\,m_2+1,\,m_3\rangle \\
S_+^{13}\,|m_1,\,m_2,\,m_3\rangle &= \sqrt{m_1(m_3+1)}\,|m_1-1,\,m_2,\,m_3+1\rangle \\
S_-^{12}\,|m_1,\,m_2,\,m_3\rangle &= \sqrt{m_2(m_1+1)}\,|m_1+1,\,m_2-1,\,m_3\rangle \\
S_-^{13}\,|m_1,\,m_2,\,m_3\rangle &= \sqrt{m_3(m_1+1)}\,|m_1+1,\,m_2,\,m_3-1\rangle \\
S_z^{12}\,|m_1,\,m_2,\,m_3\rangle &= \frac{m_2-m_1}{2}\,|m_1,\,m_2,\,m_3\rangle \\
S_z^{13}\,|m_1,\,m_2,\,m_3\rangle &= \frac{m_3-m_1}{2}\,|m_1,\,m_2,\,m_3\rangle
\end{aligned} \tag{6.55}$$

where m_k is the number of atoms at the k–th level, so that $m_1+m_2+m_3=A$. The Hamiltonian Equation 6.49 admits two integrals of motion, which can be interpreted as the excitation-number operators in each mode:

$$N_1 = a_1^\dagger a_1 + S_z^{12} + \frac{A}{2}, \quad N_2 = a_2^\dagger a_2 + S_z^{13} + \frac{A}{2}, \quad [N_1, H] = [N_2, H] = 0$$

which means that: $m_2+n_1=\text{const}$ and $m_3+n_2=\text{const}$, where n_1 and n_2 are numbers of photons in the field modes.

We start considering three-level atoms in the resonant classical fields (in the rotating frame with respect to each field, after application of the RWA), so that, in Equation 6.50, we treat a, $a^\dagger$ as c-numbers

$$H_{\text{cl}} = g_1\left(S_+^{12}+S_-^{12}\right) + g_2\left(S_+^{13}+S_-^{13}\right) \tag{6.56}$$

In this case, the Hamiltonian can easily be diagonalized by an appropriate unitary transformation from the $su(3)$ group. It is convenient to look for such a transformation in the normal form (Gauss decomposition)

$$w = \mathrm{e}^{\zeta S_-^{23}}\mathrm{e}^{\chi S_-^{13}}\mathrm{e}^{\xi S_-^{12}}\mathrm{e}^{\ln\delta_1 2S_z^{12}}\mathrm{e}^{\ln\delta_2 2S_z^{13}}\mathrm{e}^{\alpha S_+^{12}}\mathrm{e}^{\beta S_+^{13}}\mathrm{e}^{\gamma S_+^{23}} \tag{6.57}$$

The coefficients in the above decomposition can be found using the smallest faithful representation of $su(3)$, since the transformation rules depend only on the commutation relations of the algebra and not on the matrix elements of the operators in a particular representation. Thus, using the fundamental three-dimensional representation, we obtain the Hamiltonian

$$H_{\text{cl}} = \begin{bmatrix} 0 & 0 & g_2 \\ 0 & 0 & g_1 \\ g_2 & g_1 & 0 \end{bmatrix} = w\begin{bmatrix} \Omega & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\Omega \end{bmatrix} w^\dagger$$

which is diagonalized by the operator

$$w = \frac{1}{\sqrt{2}\Omega}\begin{bmatrix} g_2 & \sqrt{2}g_1 & g_2 \\ g_1 & -\sqrt{2}g_2 & g_1 \\ \Omega & 0 & -\Omega \end{bmatrix}, \quad \Omega = \sqrt{g_1^2+g_2^2} \tag{6.58}$$

Using explicit 3 × 3 matrix expression for each piece of the Gauss decomposition (6.57), we obtain w in the form

$$w = \begin{bmatrix} 1 & 0 & 0 \\ \zeta & 1 & 0 \\ \chi & \xi & 1 \end{bmatrix} \begin{bmatrix} \delta_2 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_1^{-1}\delta_2^{-1} \end{bmatrix} \begin{bmatrix} 1 & \gamma & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix} \tag{6.59}$$

Direct comparison of Equations 6.58 and 6.59 gives the following solution for the Gaussian parameters:

$$\delta_1 = \frac{g_2}{\sqrt{2}\Omega}, \quad \delta_2 = \frac{-\Omega}{g_2}, \quad \alpha = 0, \quad \beta = 1; \quad \gamma = \frac{\sqrt{2}g_1}{g_2}$$

$$\zeta = \frac{g_1}{g_2}, \quad \chi = \frac{\Omega}{g_2}, \quad \xi = \frac{g_1}{\Omega}$$

Since Equation 6.57 is valid in any representation, one may compute matrix elements of the transformation $\langle m_1,\ m_2,\ m_3|w|m_1,\ m_2,\ m_3\rangle$. Note, that $\ln\delta_{1,2}$ actually do not appear in the final expression, and the diagonal factors in Equation 6.57 have the following form,

$$\langle m_1,\ m_2,\ m_3|\mathrm{e}^{\ln\delta_1 2S_z^{12}}\mathrm{e}^{\ln\delta_2 2S_z^{13}}|m_1,\ m_2,\ m_3\rangle = \delta_1^{m_2-m_1}\delta_2^{m_3-m_1}$$

Using the parameterization (6.57) it is easy to find, e.g. the eigenvectors of the Hamiltonian (6.56),

$$\left|\underline{m_1,\ m_2,\ m_3}\right\rangle = w\,|m_1,\ m_2,\ m_3\rangle \tag{6.60}$$

We recall that $|m_1, m_2, m_3\rangle$, is a bare state of the atomic system (the same basis as in Equation 6.55).

On the other hand, using three-dimensional realization of the atomic operators (6.54) and the explicit form of the rotation matrix (6.58), one can immediately find the transformation of an arbitrary operator from the $su(3)$ algebra under the $su(3)$ rotation w. For instance, we have

$$w^\dagger S_x^{12} w = \frac{g_1}{\Omega} S_z^{13} + \frac{g_2}{\sqrt{2}\Omega}\left(S_x^{12} - S_x^{23}\right)$$

$$w^\dagger S_x^{13} w = \frac{g_2}{\Omega} S_z^{13} - \frac{g_1}{\sqrt{2}\Omega}\left(S_x^{12} - S_x^{23}\right)$$

where $S_x^{ij} = (S_+^{ij} + S_-^{ij})/2$. Therefore, the Hamiltonian is transformed as follows:

$$w^\dagger H_{\mathrm{cl}} w = w^\dagger\left(g_1 2S_x^{12} + g_2 2S_x^{13}\right) w = 2\Omega S_z^{13} \tag{6.61}$$

which is obviously independent of the representation. It follows from Equation 6.56 that

1. the spectrum of (6.56) is equidistant; the Hamiltonian acts on the eigenvectors (6.60), as

$$H_{\mathrm{cl}}\left|\underline{m_1,\ m_2,\ m_3}\right\rangle = \Omega(m_3 - m_2)\left|\underline{m_1,\ m_2,\ m_3}\right\rangle$$

2. the Hamiltonian (6.56) is an element of the $su(2)$ subalgebra of the $su(3)$ algebra. The irreducible representation of the full $su(3)$ algebra is reducible under the action of the $su(2)$ subalgebra, so that the spectrum is degenerate. Considering the action of the $su(2)$ subalgebra S_z^{13}, $S_\pm^{13}$ in the basis (6.55), we notice that this subalgebra does not change the index m_2. Therefore, the irreducible blocks (under action of the $su(2)$ subalgebra) correspond to different values of m_2: $m_2 = 0, 1, 2, \ldots, A$, and for a given value of m_2, the block is $A - m_2 + 1$-dimensional. The total dimension of the $su(3)$ algebra representation is then

$$(A+1) + (A) + (A-1) + \cdots + 1 = \frac{(A+2)(A+1)}{2}$$

as it should be. Note that even the simplest three-dimensional representation of $su(3)$ is reducible under the action of the $su(2)$ subalgebra: the three-dimensional representation decomposes into a direct sum of a two-dimensional and a one-dimensional representations.

Finally, we briefly describe how to develop the perturbation theory for the quantum field case. We denote by $|n_j\rangle_{fj}$ the state of j th field mode with n_j photons. The interaction Hamiltonian (6.50) in the basis of the total system space

$$\begin{aligned} &|m_2,\ m_3\rangle = |A - m_2 - m_3,\ m_2,\ m_3\rangle_{\mathrm{a}}\, |N_1 - m_2\rangle_{f1} |N_2 - m_3\rangle_{f2} \\ &m_2 = 0, 1, 2, \ldots, A;\ \ m_3 = 0, 1, \ldots, A - m_2 \end{aligned}$$

has the following nonvanishing matrix elements:

$$\begin{aligned} \langle m_2 + 1, m_3|H|m_2, m_3\rangle &= g_1\sqrt{(m_2+1)(A - m_2 - m_3)(N_1 - m_2)} \\ \langle m_2 - 1, m_3|H|m_2, m_3\rangle &= g_1\sqrt{m_2(A - m_2 - m_3 + 1)(N_1 - m_2 + 1)} \\ \langle m_2, m_3 + 1|H|m_2, m_3\rangle &= g_2\sqrt{(m_3+1)(A - m_2 - m_3)(N_2 - m_3)} \\ \langle m_2, m_3 - 1|H|m_2, m_3\rangle &= g_2\sqrt{m_3(A - m_2 - m_3 + 1)(N_2 - m_3 + 1)} \end{aligned} \quad (6.62)$$

By analogy with the $su(2)$ case, we define the zeroth-order approximation as follows;

$$H_0 = g_1\sqrt{N_1 - \frac{A}{2} + \frac{1}{2}}\left(S_+^{12} + S_-^{12}\right) + g_2\sqrt{N_2 - \frac{A}{2} + \frac{1}{2}}\left(S_+^{13} + S_-^{13}\right)$$

Introducing the small parameters

$$\epsilon_j = \frac{1}{\left(N_j - \dfrac{A}{2} + \dfrac{1}{2}\right)}, \quad j = 1, 2$$

and expanding the last factors in every square root in Equation 6.62, we have

$$H = H_0 + H_1 + H_2 + \cdots$$

where nonvanishing matrix elements of the Hamiltonians H_1 (in a subspace with given numbers of excitations) are

$$\begin{aligned}
\langle m_2+1, m_3|H_1|m_2, m_3\rangle &= \frac{\sqrt{\epsilon_1}g_1}{2}\sqrt{(m_2+1)(A-m_2-m_3)}\left(m_2-\frac{A+1}{2}\right)\\
\langle m_2, m_3+1|H_1|m_2, m_3\rangle &= \frac{\sqrt{\epsilon_2}g_2}{2}\sqrt{(m_3+1)(A-m_2-m_3)}\left(m_3-\frac{A+1}{2}\right)\\
\langle m_2, m_3|H_1|m_2+1, m_3\rangle &= \langle m_2+1, m_3|H_1|m_2, m_3\rangle^*\\
\langle m_2, m_3|H_1|m_2, m_3+1\rangle &= \langle m_2, m_3+1|H_1|m_2, m_3\rangle^*
\end{aligned}$$

Note, that the zeroth-order approximation is determined by the transformations (6.57) and (6.58), where we need to replace

$$g_1 \text{ by } \tilde{g}_1 = g_1\sqrt{N_1-\frac{A}{2}+\frac{1}{2}}, \quad g_2 \text{ by } \tilde{g}_2 = g_2\sqrt{N_2-\frac{A}{2}+\frac{1}{2}}$$

6.9
Problems

6.1 For a system of A two-level atoms, the Hilbert space is a direct sum of the subspaces corresponding to the irreducible representations of the $su(2)$ algebra. For the case of four atoms, $A=4$, find the vacuum vectors of all irreducible representations, under the additional conditions that these vectors are the eigenvectors of the cyclic permutation of the atoms, Equation 6.8. Find the basis of all these representations.

Answer: One five-dimensional representation, with the vacuum vector

$$|0\rangle = |0000\rangle \tag{6.63}$$

three three-dimensional representations, with the vacuum vectors

$$|1,1\rangle = \frac{1}{2}\left(|1000\rangle + \mathrm{i}|0100\rangle - |0010\rangle - \mathrm{i}|0001\rangle\right) \tag{6.64}$$

$$|1,2\rangle = \frac{1}{2}\left(|1000\rangle - |0100\rangle + |0010\rangle - |0001\rangle\right) \tag{6.65}$$

$$|1,3\rangle = \frac{1}{2}\left(|1000\rangle - \mathrm{i}|0100\rangle - |0010\rangle + \mathrm{i}|0001\rangle\right) \tag{6.66}$$

(6.67)

and two one-dimensional representations

$$|2,1\rangle = \frac{1}{\sqrt{12}}\big(|1100\rangle + |0011\rangle + |1001\rangle + |0110\rangle - 2|0101\rangle - 2|1010\rangle\big) \tag{6.68}$$

$$|2,2\rangle = \frac{1}{2}\left(|1100\rangle + |0011\rangle - |1001\rangle - |0110\rangle\right) \tag{6.69}$$

6.2 Find the exact eigenvalues and eigenvectors for the Dicke model Hamiltonians (6.12), (6.13) under the exact resonance condition for the case of two atoms, $A = 2$. Compare with the results in the first order of the perturbation theory (6.27), (6.31).

6.3 Prove that the components of the zeroth-order eigenvectors (6.23) have the following symmetry properties:

(a) $\alpha^A_{kp} = \alpha^A_{pk}$,

(b) $\alpha^A_{pk} = (-1)^p \alpha^A_{p,A-k}$,

(c) $\alpha^A_{pk} = (-1)^k \alpha^A_{A-p,k}$,

(d) $\alpha^A_{pk} = (-1)^{k+p-A} \alpha^A_{A-p,A-k}$.

6.4 Analyze the evolution of the initial squeezed atomic state (1.29) in the Dicke model (6.12), (6.13) if the initial state of the field is

(a) a number state $|N\rangle$ with $N \ll A$;

(b) a coherent state $|\alpha\rangle$ with $\overline{n} = |\alpha|^2 \ll A$.

6.5 For the Dicke model with the case of finite detuning, $\Delta = \omega_f - \omega_a$, develop the perturbation theory analogous to (6.27), (6.31). Find the eigenvalues in the second order and the eigenvectors in the first order. Hint: first, apply to the Hamiltonian

$$H = \omega \hat{N} + \Delta S_z + g\left(a S_+ + S_- a^\dagger\right)$$

the transformation, $e^{i\phi S_y}$.

6.6 Consider A identical three-level atoms with V-configuration of the levels.

(a) For two and three atoms, $A = 2, 3$ find explicitly the decomposition of the atomic Hilbert space into a direct sum of irreducible representations of $su(3)$. For each irreducible representation, find the basis and the matrix elements of all of the operators. Hint: the lowest states of the irreducible representations are annihilated by the lowering operators, $S^{km}_-|0\rangle = 0$; the highest states are annihilated by the rising operators, $S^{km}_+|h\rangle = 0$; in each irreducible representation, look for basis vectors in the form $\prod S^{jk}_+|0\rangle$.

(b) Find the basis and the matrix elements for the fully symmetrical representation for arbitrary A.

(c) Consider A three-level atoms of V-configuration and transition frequencies ω_{12}, ω_{13}, interacting with the two modes of quantized field with frequencies ω_{f1}, ω_{f2}, under the exact resonance conditions: $\omega_{12} = \omega_{f1}$, $\omega_{13} = \omega_{f2}$, so that the Hamiltonian has a form (6.50). For the case of weak field, $A \gg N$, develop the perturbation theory analogous to the one of Sections 6.4 and 6.5 and find the eigenvalues of the Hamiltonian in the second order.

7
Atomic Systems in a Strong Quantum Field

The results of Chapter 5 for a single atom interacting with a strong quantum field can be generalized to the case of an arbitrary atomic system provided that the number of excitations in the atomic system is much smaller than the number of photons in the field mode. The method we are going to use is a perturbation theory over the inverse field intensity developed in Chapter 6. As in Chapter 5, we introduce the semiclassical atomic Hamiltonian. Its eigenstates play a special role; being taken as initial atomic states they lead to the approximate factorization of the total wave function into the field and atomic parts. Evolution of an arbitrary initial atomic state is described by a superposition of the factorized states.

7.1
Dicke Model in a Strong Field

We start with the generalization of the operator $\hat{Q}$ introduced in Chapter 5 for a single atom case (Equations 5.12 and 5.13), to the case of A two-level atoms [46]. We consider the following transformation:

$$\hat{Q} = \exp\left[\mathrm{i}\hat{\phi}\left(S_z + \frac{A}{2}\right)\right] \tag{7.1}$$

Here $\mathrm{e}^{\mathrm{i}\hat{\phi}}$ is the field phase operator (3.126) and (3.127). Since the operator $S_z + A/2$ has an integer spectrum, the operator $\hat{Q}$ is a direct sum of powers of the phase operator

$$\hat{Q} = \sum_{k=0}^{A} \mathrm{e}^{\mathrm{i}k\hat{\phi}} |k\rangle_{\mathrm{at}\ \mathrm{at}}\langle k|$$

The action of operator $\hat{Q}$ on the vectors of the basis (6.19) (where $|k\rangle_{\mathrm{at}}$ is an atomic state with k atoms excited and $|n\rangle_f$ is a field state with n photons) is given by

$$\begin{aligned}\hat{Q}|N,k\rangle &= \hat{Q}|k\rangle_{\mathrm{at}} \otimes |n = N-k\rangle_f = \mathrm{e}^{\mathrm{i}\hat{\phi}k}\,|k\rangle_{\mathrm{at}} \otimes |n\rangle_f \\ &= |k\rangle_a \otimes |n-k\rangle_f = |n,k\rangle\end{aligned}$$

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We stress that the operator $\hat{Q}$ satisfies the commutation relation $[\hat{Q}, \hat{Q}^\dagger] = 0$ and is unitary only on the states $|n, k\rangle$ with $n > A \geq k$. However, this is sufficient for our goals.

The following commutation relations for the operator $\hat{Q}$ can be found from the properties of the phase operator:

$$f(\hat{n})\hat{Q}^\dagger = \hat{Q}^\dagger f\left(\hat{n} + S_z + \frac{A}{2}\right), \quad \hat{Q} f(\hat{n}) = f\left(\hat{n} + S_z + \frac{A}{2}\right)\hat{Q} \tag{7.2}$$

From Equation 7.2 we conclude that $\hat{Q}$ transforms the photon number into the excitation number (up to the constant term): $\hat{N} = \hat{Q}\left(\hat{n} - A/2\right)\hat{Q}^\dagger$. Applying the transformation $\hat{Q}$ to the atomic and field operators, we obtain

$$\hat{Q}S_+\hat{Q}^\dagger = \exp(\mathrm{i}\hat{\phi})S_+, \quad \hat{Q}S_-\hat{Q}^\dagger = \exp(-\mathrm{i}\hat{\phi})S_- \tag{7.3}$$

$$\hat{Q}^\dagger a \hat{Q} = \sqrt{\hat{n} - S_z - \frac{A}{2} + 1}\,\mathrm{e}^{\mathrm{i}\hat{\phi}} = \sqrt{\frac{\hat{n} - S_z - \frac{A}{2} + 1}{\hat{n} + 1}}\, a \tag{7.4}$$

Now let us apply the $\hat{Q}$ transformation to the Hamiltonian of the Dicke model (6.12)

$$H = \omega\hat{N} + \Delta S_z + V, \quad \hat{N} = \hat{n} + S_z, \quad V = g\left(aS_+ + a^\dagger S_-\right) \tag{7.5}$$

Using Equations 7.2– 7.4 we find that $\hat{Q}$ transformation diagonalizes the Hamiltonian in the field space

$$H_{\mathrm{at}}\left(\hat{n}\right) = \hat{Q}^\dagger H \hat{Q}$$

where

$$H_{\mathrm{at}}\left(\hat{n}\right) \equiv \omega\hat{\upsilon} + \Delta S_z + g\left(\sqrt{\hat{\upsilon} - S_z + 1}S_+ + S_-\sqrt{\hat{\upsilon} - S_z + 1}\right) \tag{7.6}$$

$$\hat{\upsilon} = \hat{n} - \frac{A}{2}$$

The price that we have to pay for eliminating the nondiagonal field operators is the nonlinearity of the atomic Hamiltonian H_{at}.

This approach is useful if we can express the Hamiltonian H_{at} as a linear function of the generators of some representation of a certain finite-dimensional Lie algebra. This is approximately the case when the initial field is strong, that is, when the number of excitations is much larger than the maximum number of atomic excitations ($\hat{N} \sim \hat{n} \gg S_z$). In this case, it is possible to develop a perturbation theory, with the inverse of the field intensity as a small parameter. To achieve this goal, we expand the square roots in Equation 7.6 as a power series taking $(\hat{\upsilon} + 1/2)^{-1/2}$ as a small parameter. This yields

$$H_{\mathrm{at}} = H_0 + \frac{1}{\sqrt{\hat{\upsilon} + 1/2}}H_1 + O\left(\hat{\upsilon}^{-3/2}\right) \tag{7.7}$$

where

$$H_0 = \omega\hat{\upsilon} + \Delta S_z + g\sqrt{\hat{\upsilon} + \frac{1}{2}}\,(S_+ + S_-) \tag{7.8}$$

$$H_1 = -\frac{1}{4}g\,\{S_z, S_+ + S_-\}_+ \tag{7.9}$$

$\{,\}_+$ is the anticommutator. Choosing of the expansion parameter in the form $(\hat{\upsilon} + 1/2)^{-1/2}$ allows us to eliminate the first-order corrections to the eigenfrequencies in the resonant case. Indeed, if $\Delta = 0$, one can show that the average of H_1 over the eigenstates of H_0 is zero. The Hamiltonian $\hat{Q}H_0\hat{Q}^\dagger$ has a transparent physical sense: in each subspace with a given excitation number N, the dynamics is approximately linear and corresponds to a rotation of the Bloch vector around the x axis. The rotation frequency depends in a nonlinear way on the subspace index N, which reflects the nonlinearity of the model. (For instance, in the case $\Delta = 0$, the rotation frequency is $\Omega_N = g\sqrt{N - A/2 + 1/2}$.) We call H_0 the *quasi-linear approximation*. The zeroth-order evolution operator $U_0(t)$

$$U_0(t) = \hat{Q}\,U_0^{(\mathrm{at})}(t)\hat{Q}^\dagger, \quad U_0^{(\mathrm{at})}(t) = \exp(-\mathrm{i}tH_0) \tag{7.10}$$

is an element of the $su(2)$ group (since the Hamiltonian $\hat{Q}H_0\hat{Q}^\dagger$ is an element of the $su(2)$ algebra) and its matrix elements are easily obtained using the group representation theory. It is easy to see that in the case of one atom, $A = 1$, the evolution operator U_0 is reduced to the exact evolution operator for the Jaynes–Cummings model (JCM) (Equation 5.5).

The first-order correction to the evolution operator (7.10) can also be found using, for example, the following formula,

$$\mathrm{e}^{-\mathrm{i}(A+\varepsilon B)t} = \mathrm{e}^{-\mathrm{i}At}\left[1 - \mathrm{i}\varepsilon\int_0^t \mathrm{d}\tau \mathrm{e}^{\mathrm{i}A\tau}B\mathrm{e}^{-\mathrm{i}A\tau} + O\left(\varepsilon^2\right)\right]$$

Defining $A = H_0$, $B = H_1$, in the case of exact resonance, we obtain

$$U^{(\mathrm{at})}(t) \approx U_0^{(\mathrm{at})}(t)(1 + R_1 + R_2 + \cdots)$$

$$R_1 = \frac{\mathrm{i}}{4g^2\left(\hat{\upsilon} + 1/2\right)}\left\{S_x, S_z\sin\hat{\vartheta} - S_y\left(1 - \cos\hat{\vartheta}\right)\right\}$$

where

$$\hat{\vartheta} = 2gt\sqrt{\hat{\upsilon} + \frac{1}{2}}$$

One can see that R_1 does not increase with time and is of the order $1/\overline{n}$. The second-order corrections increase with time and are of the order

$$R_2 \sim \frac{gt}{\overline{n}^{3/2}}$$

which means that in the resonant case, the evolution operator in the form (7.10) describes the dynamic behavior of the system quite well (with an error in the coefficients of order $A^2/\overline{n}$, where A is the number of atoms) for the times $gt \leq \overline{n}^{3/2}$.

Using the evolution operator in the form (7.10), we can find the evolution for the system operators in the Heisenberg representation. For example, since $[S_z, \hat{Q}] = 0$, the Heisenberg operator $S_z(t)$ is given by

$$S_z(t) = \hat{Q} U_{\text{at}}^{\dagger} S_z U_{\text{at}} \hat{Q}^{\dagger} = \hat{Q} \left(S_z \cos \hat{\vartheta} + S_y \sin \hat{\vartheta} \right) \hat{Q}^{\dagger} \tag{7.11}$$

or, finally,

$$S_z(t) = S_z \cos \hat{\xi} + \frac{1}{2\mathrm{i}} \left(S_+ \mathrm{e}^{\mathrm{i}\hat{\phi}} \sin \hat{\xi} - \sin \hat{\xi} \mathrm{e}^{-\mathrm{i}\hat{\phi}} S_- \right)$$

where

$$\hat{\xi} = \hat{Q} \hat{\vartheta} \hat{Q}^{\dagger} = 2gt \sqrt{\hat{n} + \frac{1}{2} + S_z}$$

In the case of one atom, the expression (7.11) coincides with the exact result of Equation 5.9.

7.2
Factorization of the Wave Function

As in the JCM , the Dicke model has the following interesting feature: there are some initial states for which the wave function of the system is approximately written as a product of pure states of the field and of the atomic system [83]. In the case of the JCM, the factorization of the wave function is obtained by using the exact evolution operator (see Chapter 5). In the case of the Dicke Model, in order to prove the factorization we use the quasi-linear approximation.

Let us assume that at $t = 0$ the field is taken in a coherent state $|\alpha\rangle$, where $\alpha = \sqrt{\overline{n}} \mathrm{e}^{\mathrm{i}\varphi}$ and the initial number of photons is large, $\overline{n} \gg A$. It is useful to introduce the atomic system Hamiltonian in a classical field,

$$H_{\text{cl}} = \mathrm{e}^{\mathrm{i}\varphi} S_+ + \mathrm{e}^{-\mathrm{i}\varphi} S_- \tag{7.12}$$

where the classical field has the same phase φ as the initial field coherent state. The eigenstates of the classical field Hamiltonian (7.12) (“*semiclassical eigenstates*”) are

$$H_{\text{cl}} |\underline{p, \varphi}\rangle = \lambda_p |\underline{p, \varphi}\rangle$$

where $\lambda_p = 2p - A$ and $p = 0, 1, \ldots A$. It is clear that

$$|\underline{p, \varphi}\rangle = \mathrm{e}^{\mathrm{i}\varphi(S_z + A/2)} |\underline{p}\rangle$$

where the vectors $|\underline{p}\rangle$ are the eigenvectors of the operator S_x where

$$2S_x |\underline{p}\rangle = \lambda_p |\underline{p}\rangle \tag{7.13}$$

We take one of the eigenstates of the classical field Hamiltonian 7.12 as the initial state of the atomic system. Therefore, the initial state of the total system is

$$|\Psi_p(0)\rangle = |\alpha\rangle \otimes |\underline{p,\varphi}\rangle \tag{7.14}$$

Applying the evolution operator in the form (7.10) to the initial state (7.14) and taking into account that coherent states are almost the eigenstates for the phase operator

$$\hat{Q}^{\dagger}|\alpha\rangle = \mathrm{e}^{-\mathrm{i}\hat{\phi}(S_z+A/2)}|\alpha\rangle = \mathrm{e}^{-\mathrm{i}\varphi(S_z+A/2)}|\alpha\rangle + O\left(\frac{1}{\sqrt{\overline{n}}}\right) \tag{7.15}$$

we obtain

$$|\Psi(t)\rangle = U_0(t)|\Psi_p(0)\rangle \approx \hat{Q}\mathrm{e}^{-2\mathrm{i}tg\sqrt{\hat{\nu}+1/2}S_x}|\alpha\rangle|\underline{p}\rangle$$

Since the state $|\underline{p}\rangle$ is an eigenstate of the operator S_x, we have

$$|\Psi(t)\rangle \approx \hat{Q}\mathrm{e}^{-\mathrm{i}tg\sqrt{\hat{\nu}+1/2}\,\lambda_p}|\alpha\rangle|\underline{p}\rangle$$

Using the commutation rule

$$\hat{Q}\mathrm{e}^{-\mathrm{i}tg\sqrt{\hat{\nu}+1/2}\,\lambda_p} = \mathrm{e}^{-\mathrm{i}tg\sqrt{\hat{\nu}+S_z+A/2+1/2}\,\lambda_p}\hat{Q}$$

we obtain

$$|\Psi(t)\rangle \approx \mathrm{e}^{-\mathrm{i}tg\sqrt{\hat{\nu}+S_z+A/2+1/2}\,\lambda_p}|\Psi_p(0)\rangle \tag{7.16}$$

Now we expand the frequency in the exponential in Equation 7.16 in a Taylor series,

$$\sqrt{\hat{\nu}+S_z+\frac{A}{2}+\frac{1}{2}} \simeq \sqrt{\hat{\nu}+\frac{1}{2}} + \frac{S_z+A/2}{2\sqrt{\overline{\nu}+1/2}} + O\left(\frac{A^2}{\overline{n}^{3/2}}\right)$$

We obtain

$$|\Psi(t)\rangle = |\Phi_p(t)\rangle_f \otimes |A_p(t)\rangle_{\mathrm{at}} \tag{7.17}$$

$$|\Phi_p(t)\rangle_f = \mathrm{e}^{-\mathrm{i}gt\sqrt{\hat{\nu}+1/2}\,\lambda_p}|\alpha\rangle \tag{7.18}$$

$$|A_p(t)\rangle_{\mathrm{at}} = \exp\left(-\mathrm{i}gt\frac{S_z+A/2}{2\sqrt{\overline{\nu}+1/2}}\,\lambda_p\right)|\underline{p,\varphi}\rangle \tag{7.19}$$

where $\overline{\nu} = \overline{n} - A/2$. It can easily be seen that for the case of one atom, $A = 1$, the results obtained for the JCM are reconstructed. The factorization (Equation 7.17) is valid for the times $gt < \sqrt{\overline{n}}$ (due to the approximation in frequencies) with an accuracy in the amplitudes of the order $O(A/\sqrt{\overline{n}})$ (due to the approximation, Equation 7.15). Note that the photon distribution is always Poissonian. Of course (as in the case of the JCM), this factorization does not mean that the systems evolve in an independent way.

From the factorization (7.17), it immediately follows that (as in the JCM) the states $|\Psi_p(0)\rangle = |\alpha\rangle \otimes |\underline{p,\varphi}\rangle$; $p = 0, 1, \ldots, A$ are trapping states:

$$\langle\Psi_p(t)|S_z|\Psi_p(t)\rangle = \langle\Psi_p(0)|U^{\dagger}S_zU|\Psi_p(0)\rangle \simeq \langle\Psi(0)|S_z|\Psi(0)\rangle$$

Thus, the atomic inversion practically does not evolve for these initial states. The evolution of the atomic inversion for the initial atomic trapping state is shown in Figure 7.1 (the graph was obtained by the exact numerical calculation with the Dicke model Hamiltonian). Small oscillations seen in the figure represent corrections to the analytic approximation of this chapter.

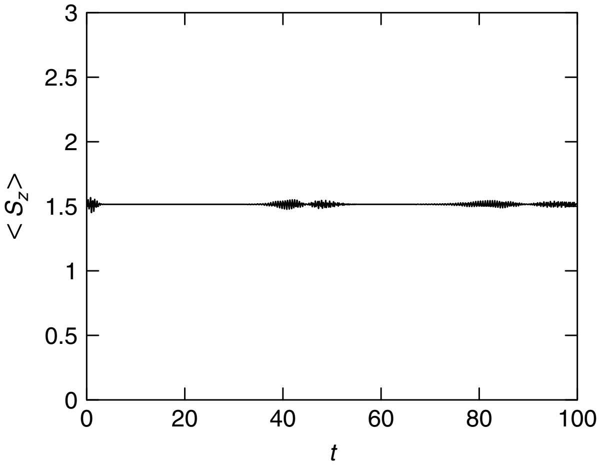

Figure 7.1 Atomic inversion dynamics: the field is taken initially in its coherent state with $\overline{n} = 49$, the atomic system, $A = 3$, is initially in the trapping state (the eigenstate of the classical field Hamiltonian) $|\underline{p = 0,\ \phi = 0}\rangle_a$.

7.3
Evolution in Phase Space

It follows from Equation 7.18, that the evolution of the field in the factorized wave function is described by an effective Hamiltonian H_{eff} [83]

$$H_{\text{eff}} = g\lambda_p\sqrt{\hat{n} - \frac{A}{2} + \frac{1}{2}} \tag{7.20}$$

Expanding the square root in the equation above in the powers of $\widehat{\Delta n} \equiv \hat{n} - \overline{n} \sim \sqrt{\overline{n}}$ we obtain

$$H_{\text{eff}} = g\lambda_p\left[\sqrt{\overline{n}_A} + \frac{\widehat{\Delta n}}{2\sqrt{\overline{n}_A}} - \frac{(\widehat{\Delta n})^2}{8\overline{n}_A{}^{3/2}}\right] \tag{7.21}$$

where we have neglected the terms of order $t(\widehat{\Delta n})^3/\overline{n}^{5/2} \sim t/\overline{n}$. Each term in Equation 7.21 has a clear physical sense: the first term leads to the multiplication of the initial coherent state by a phase factor; the second term moves the state of the field around a circle in field phase space; finally, the third term describes the phase dispersion that leads to the deformation of the initial coherent state. This effect is similar to what happens during evolution in the Kerr medium. However, there is a substantial difference: the term that describes the dispersion has the opposite sign in the Dicke model to the one in the Kerr medium; that is, the regions that correspond to the higher intensities move more slowly. When the phase dispersion reaches 2π, self-interference effects take place (like in the Kerr medium). These self-interference effects produce phenomena similar to the "Schrödinger cats" in the Kerr medium (which in this case are called *fractional revivals*) [59]. However, since the Dicke model spectrum contains noncommensurate frequencies the fractional revivals are not as pronounced as the cats in the Kerr medium.

For short times $gt \ll \sqrt{\overline{n}}$, the nonlinear term in Equation 7.21 does not contribute significantly and the field state remains a coherent state so that

$$|\Phi_p(t)\rangle_f \approx \mathrm{e}^{-\mathrm{i}g\lambda_p t\sqrt{\overline{n}-A/2+1/2}/2}|\mathrm{e}^{-\mathrm{i}\omega_p t}\alpha\rangle \tag{7.22}$$

where

$$\omega_p = \frac{g\lambda_p}{2\sqrt{\overline{n}-A/2+1/2}} \tag{7.23}$$

using the fact that

$$\mathrm{e}^{-\mathrm{i}\omega_p t\hat{n}}|\alpha\rangle = |\mathrm{e}^{-\mathrm{i}\omega_p t}\alpha\rangle$$

This means that for short times (before the first revival) the initial coherent state of the field simply rotates in the phase plane with the frequency ω_p. To represent the evolution in the case of an initial semiclassical eigenstate $|\underline{p}\rangle$ in the phase plane (for simplicity we take the initial phase of the field coherent state to be equal to zero: $\varphi = 0$), we use the field quasiprobability distribution $Q(\alpha)$. Under the approximation of the factorized states (7.17) to (7.19), the Q function is reduced to

$$Q_p\,(\beta) = |\langle\beta|\Phi_p(t)\rangle_f|^2$$

In the beginning of the evolution, when Equation 7.22 is satisfied, the function Q has the form of a hump (whose contour lines are circles) with the center moving with the angular velocity ω_p

$$Q\,(\beta) = |\langle\beta|\mathrm{e}^{-\mathrm{i}\omega_p t}\alpha\rangle_f|^2 = \exp\left[-|\beta - \mathrm{e}^{-\mathrm{i}\omega_p t}\alpha|^2\right]$$

For longer times, the shape of the function Q is deformed, but its center continues to move as in the beginning. The function Q corresponding to the state $|\Phi_p(t)\rangle$ can be represented in the following form (this expression has been obtained by using Equation 7.21):

$$Q_p(\beta = r\,\mathrm{e}^{\mathrm{i}\phi}) = \frac{\exp\left[-(r-r_0)^2\right]}{\sqrt{rr_0\mu(t)}} \sum_{k=-\infty}^{+\infty} \exp\left[-\frac{(\phi + \phi_0(t) - 2\pi k)^2}{\mu(t)}\right] \tag{7.24}$$

Each term in the sum is a Gaussian function (in phase) whose center moves with a constant angular velocity, $\phi_0(t) = tg\lambda_p/(2\sqrt{rr_0 - A/2 + 1/2})$, and whose variance depends on the intensity

$$\mu(t) = \frac{1}{rr_0} + \left(\frac{g\lambda_p t}{2rr_0}\right)^2, \quad r_0 \equiv \sqrt{\overline{n}}$$

that describes a phase diffusion. Each term in the sum (7.24) contributes at different times. For example, for the times $gt \le 2\pi\sqrt{\overline{n}}/\lambda_p$, only the term with $k = 0$ contributes effectively in the sum. Thus, the contour lines of the Q function are given as

$$(r - r_0)^2 + \frac{(\phi - \phi_0(t))^2}{\mu(t)} = \text{const}$$

Now, let us consider the evolution of an arbitrary atomic state in a strong coherent field. Any initial atomic state can be expanded in the semiclassical basis $|\underline{p}\rangle$ (we choose the initial field phase $\varphi = 0$) as

$$|\psi(0)\rangle_{\text{at}} = \sum_{p=0}^{A} c_p |\underline{p}\rangle, \quad c_p = \langle \underline{p}|\psi(0)\rangle_{\text{at}} \tag{7.25}$$

Thus, the state of the total system is represented as a superposition of the factorized states

$$|\Psi\,(t)\rangle = \sum_{p=0}^{A} c_p |\Phi_p\,(t)\rangle_f \otimes |A_p\,(t)\rangle_{\text{at}} \tag{7.26}$$

This means that a generic atomic state produces $A + 1$ humps in the phase plane of the field. Each hump rotates along the circle of radius $\sqrt{\overline{n}}$ with angular velocity ω_p (Equation 7.23), see Figure 7.2. These humps can collide with each other in phase space. To understand the effect of such a collision, we calculate the atomic inversion for an arbitrary initial atomic state (7.25) using the factorization approximation (Equation 7.17). From the expansion (7.26), we have

$$\langle\Psi(t)|S_z|\Psi(t)\rangle \approx \sum_{p,q=0}^{A} c_p c_q^*\,{}_{\text{at}}\langle A_q(t)|S_z|A_p(t)\rangle_{\text{at}\,f}\langle\Phi_q(t)|\Phi_p(t)\rangle_f \tag{7.27}$$

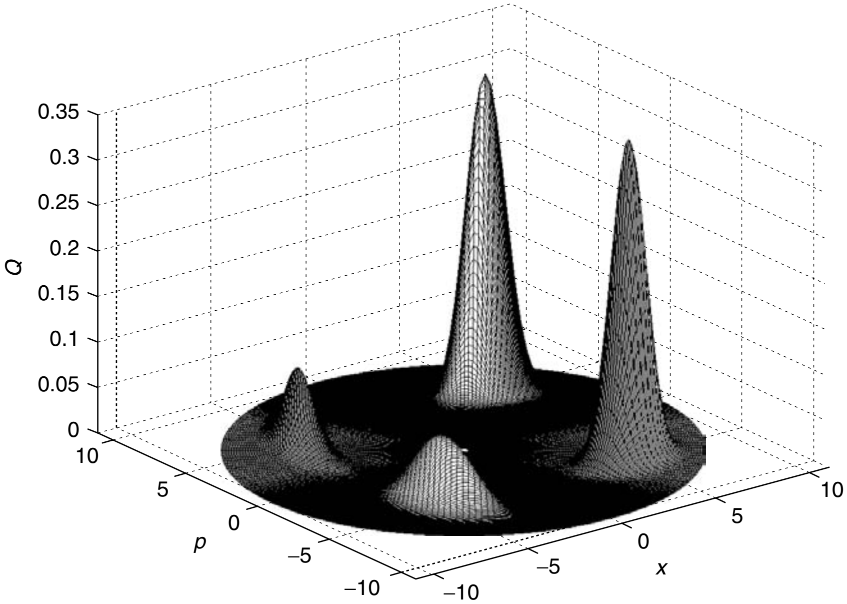

Figure 7.2 Field Q function for three atoms at the time instant $T_r/4$; the field initially prepared in a coherent state with $\overline{n} = 49$, and atoms initially excited.

It can easily be seen that

$$_{\mathrm{at}}\langle A_q(t)|S_z|A_p(t)\rangle_{\mathrm{at}} = \langle \underline{q}|S_z \exp\left[-\mathrm{i}(\omega_p-\omega_q)\left(S_z+\frac{A}{2}\right)t\right]|\underline{p}\rangle \tag{7.28}$$

The field overlap integral

$$_f\langle \Phi_q(t)|\Phi_p(t)\rangle_f = \sum_n P_n \mathrm{e}^{-\mathrm{i}\,gt(\lambda_p-\lambda_q)\,\sqrt{n-A/2+1/2}} \tag{7.29}$$

has the form of an anharmonic series. To obtain qualitative information we analyze it using the short-time approximation (Equation 7.22)

$$_f\langle \Phi_q(t)|\Phi_p(t)\rangle_f \approx \mathrm{e}^{-\mathrm{i}gt(\lambda_p-\lambda_q)\,\sqrt{\bar{n}-A/2+1/2}/2}\langle \mathrm{e}^{-\mathrm{i}\omega_q t}\alpha|\mathrm{e}^{-\mathrm{i}\omega_p t}\alpha\rangle \tag{7.30}$$

The rapidly oscillating phase factor $\mathrm{e}^{-\mathrm{i}(\lambda_p-\lambda_q)\,gt\sqrt{\bar{n}}/2}$ in Equation 7.30 represents Rabi oscillations. The amplitude of these oscillations changes relatively slowly:

$$|\langle \mathrm{e}^{-\mathrm{i}\omega_q t}\alpha|\mathrm{e}^{-\mathrm{i}\omega_p t}\alpha\rangle| = \exp\left[\frac{-\bar{n}|\mathrm{e}^{-\mathrm{i}\omega_q t}-\mathrm{e}^{-\mathrm{i}\omega_p t}|^2}{2}\right]$$

In the beginning of the evolution, $t=0$, and the oscillation amplitude is maximum. The oscillations collapse when

$$\sqrt{\bar{n}}\,|\mathrm{e}^{-\mathrm{i}\omega_q t}-\mathrm{e}^{-\mathrm{i}\omega_p t}| \sim 1$$

Since $\omega_p \sim 1/\sqrt{\bar{n}}$, we can expand the exponentials in series and obtain the collapse time

$$T_{\mathrm{c}} \sim \frac{1}{(\omega_p-\omega_q)\sqrt{\bar{n}}}$$

On the other hand, the humps collide when

$$(\omega_p-\omega_q)\,T_{\mathrm{R}} = 2\pi$$

which determines the time of a possible revival of Rabi oscillations; also, for these times, we find that the atomic matrix element (Equation 7.28) is almost independent of t, since

$$\exp\left[-\mathrm{i}(\omega_p-\omega_q)\left(S_z+\frac{A}{2}\right)T_{\mathrm{R}}\right] = \exp\left[-\mathrm{i}2\pi\left(S_z+\frac{A}{2}\right)\right] = I$$

Thus, for times of the order of the revival time, $t \sim T_{\mathrm{R}}$

$$\begin{aligned}_{\mathrm{at}}\langle A_q(t)|S_z|A_p(t)\rangle_{\mathrm{at}} &\approx \langle \underline{q}|S_z|\underline{p}\rangle \\ &= \frac{\sqrt{(A-p)(p+1)}}{2}\,\delta_{p+1,q} + \frac{\sqrt{(A-p+1)p}}{2}\,\delta_{p-1,q}\end{aligned}$$

This means that only a series of revivals with the revival time

$$T_{\mathrm{R}} = \frac{2\pi}{g}\sqrt{\bar{n}-\frac{A}{2}+\frac{1}{2}}$$

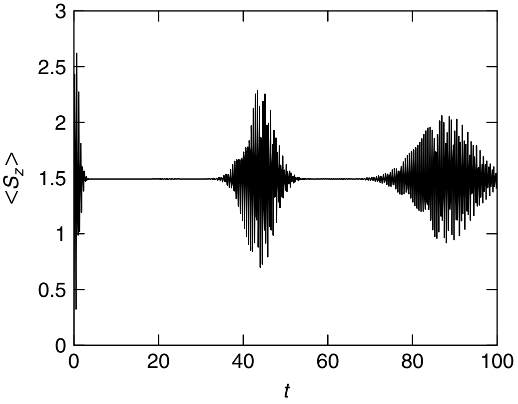

Figure 7.3 Collapses and revivals of the atomic inversion; the field initially in the coherent state with $\overline{n} = 49$, atomic system, $A = 3$ initially in its unexcited state, $|0\rangle_a$.

"survives" and the collapse time is

$$T_c = \frac{1}{g}\sqrt{\frac{\overline{n} - A/2 + 1/2}{\overline{n}}}$$

The field overlap integral will be

$$_f\langle\Phi_{p\pm1}(t)|\Phi_p(t)\rangle_f = \sum_n P_n \mathrm{e}^{\pm 2igt\sqrt{n - A/2 + 1/2}}$$

It is interesting to note that the revival time practically does not depend on the number of atoms (as long as the strong field condition is satisfied: $A \ll \overline{n}$).

Thus, the humps moving with the velocities ω_p and $\omega_{p\pm1}$ collide at time $t = T_R$. The interference of different semiclassical eigenstates is reflected in the revival effect, Figure 7.3. For example, in the case of a single atom (JCM), we have two humps that move with the frequency $\omega_1 = g/\sqrt{\overline{n}}$ in opposite directions. After each hump goes half way around, they collide and the revival occurs. In the case of two atoms, we have three humps, two of which move with the angular velocities $\omega_{0,2} = \pm 2g/\sqrt{\overline{n} - 1/2}$ and the third remains at rest at the starting point, $\omega_1 = 0$. The revival occurs when the moving humps collide with the hump that is at rest after completing a round. However, since these move twice as fast as in the case of a single atom, the revival time is practically the same (the small difference is due to the fact that the frequency $\Omega = g\sqrt{\overline{n} - A/2 + 1/2}$ depends weakly on the number of atoms).

7.4
Dicke Model in the Presence of the Kerr Medium

As another example, we consider the evolution of a collection of A two-level atoms that interact with a quantum field mode (Dicke model) inside a cavity that contains the Kerr medium [56]. We initially prepare the field in the coherent state $|\alpha\rangle$ with $\alpha = \sqrt{\overline{n}}e^{i\varphi}$, $\overline{n} \gg A$, and the atomic system in the semiclassical eigenstate $|\underline{p, \varphi}\rangle_{\text{at}}$ with the eigenvalue λ_p, so that the atomic phase φ coincides with the initial phase of the field (we consider the phase zero case $\varphi = 0$). Both the Kerr Hamiltonian (4.35) ($\sim\hat{n}^2$) and the effective Hamiltonian for the Dicke model for these initial states ($\sim\sqrt{\hat{n}}$), produce a phase diffusion that leads to the deformation of the initial state of the field. The Hamiltonian that describes both effects is of the form

$$H = \omega\hat{N} + \chi\hat{n}^2 + \Delta S_z + gV, \quad \hat{N} = \hat{n} + S_z, \quad V = aS_+ + a^\dagger S_- \tag{7.31}$$

The frequency shift Δ is chosen later on. The Kerr coupling constant χ is proportional to the third-order nonlinear susceptibility and is usually very small compared to the field–atom coupling constant. Since the most interesting regime corresponds to sufficiently large initial field intensities $\overline{n} \sim (g/\chi)^{2/3}$, it is convenient to define a new Kerr coupling constant as $\chi_0 \equiv \chi\overline{n}^{3/2}$. The term that corresponds to the Kerr interaction in the Hamiltonian (7.31) is written as

$$\chi\hat{n}^2 = \frac{\chi_0}{\overline{n}^{3/2}}\left(\hat{N}^2 - 2\hat{N}S_z + S_z^2\right) \tag{7.32}$$

The last term in the formula above is of order $1/\overline{n}^{3/2}$ and can be neglected. The principal contribution of the second term cancels out with the term ΔS_z in Equation 7.31 when

$$\Delta = 2\chi\overline{n}$$

is chosen as the frequency shift value. The contribution of the two terms $\Delta S_z - 2\chi\hat{N}S_z$ together is of order $1/\overline{n}$. Thus, only the first term in Equation 7.32 (which is of order $\sim\sqrt{\overline{n}}$) should be considered.

Applying the $\widehat{Q}$ transformation and repeating the calculations of Section 7.2 we obtain the wave function of the system in the factorized form

$$|\Psi(t)\rangle \approx |\Phi_p(t)\rangle \otimes |A_p(t)\rangle$$
$$|\Phi_p(t)\rangle = e^{-itH_f}|\alpha\rangle_f$$
$$|A_p(t)\rangle = \exp\left[it\left(\frac{4\chi_0 + g\lambda_p}{2\sqrt{\overline{n}}}\right)S_z\right]e^{-itH_a}|\underline{p}\rangle_{\text{at}}$$

where the effective Hamiltonian that describes the evolution of the field has an additional term compared to Equation 7.20, which describes the contribution of the Kerr medium:

$$H_{\text{eff}} = \chi\hat{n}^2 + g\lambda_p\sqrt{\hat{n} + 1/2} \tag{7.33}$$

Now we expand the effective Hamiltonian (7.33) in powers of $\widehat{\Delta n} \equiv \hat{n} - \overline{n} \sim \sqrt{\hat{n}}$ as we did in the case of the pure Dicke model (7.21):

$$H_{\text{eff}} \simeq g\lambda_p\sqrt{\overline{n}_A} + \chi\overline{n}^2 + \widehat{\Delta n}\left(\frac{g\lambda_p}{2\sqrt{\overline{n}_A}} + 2\chi\overline{n}\right) + (\widehat{\Delta n})^2\left(\chi - \frac{g\lambda_p}{8\overline{n}_A{}^{3/2}}\right) \tag{7.34}$$

The nonlinear part in the expression above contains two terms that appear with different signs (which is the consequence of the phase dispersion in the Dicke and Kerr mediums occurring in opposite directions). This difference in signs yields the possibility of forming nondispersive wave packets. One can see that the dispersive terms that are $\sim\widehat{\Delta n}^2$ in the Hamiltonian (7.34) cancel out between each other when the following condition is satisfied:

$$\chi = \frac{g\lambda_p}{8\bar{n}^{3/2}}$$

which determines the field intensity (for given values of the Dicke and Kerr coupling constants) when the phase dispersion decreases significantly.

7.5
Generation of the Field Squeezed States

Due to the nonlinearity of the effective Hamiltonian for the field (7.20), (7.21), the initial coherent state of the field (with $\bar{n} \gg A$) can evolve at certain time instants into a squeezed state. We will see that squeezed states are generated in the Dicke model at long times (around the revival time [58]) and at short times (close to the time of the first collapse [57]). These two possibilities are of an essentially different nature [84].

We start with the case of squeezing at the long times. We assume that the atomic system is prepared initially in its semiclassical eigenstate $|\underline{p}\rangle$. This initial state leads to the factorization of the wave function (7.17). Thus, the fluctuation of a canonical variable $q = \left(a + a^\dagger\right)/\sqrt{2}$ is given by

$$\langle\sigma_q\rangle_p = \langle\Phi_p(t)|q^2|\Phi_p(t)\rangle - \langle\Phi_p(t)|q|\Phi_p(t)\rangle^2 \qquad (7.35)$$

where the wave function of the field $|\Phi_p(t)\rangle$ is defined in Equation 7.18. Squeezing of the field state implies that $\langle\sigma_q\rangle_p < 1/4$. Since the expansion (7.21) of the effective Hamiltonian works well for the times that cover various revivals, we use this approximation to find the field operators in the Heisenberg representation, which yields

$$a(t) = e^{-i\tau}\exp\left[i\lambda_p g t\left(\frac{1+2\widehat{\Delta n}}{8\bar{n}^{3/2}}\right)\right] a$$

By calculating the average over the initial coherent state of the field, we obtain

$$\langle\hat{a}\rangle_p = \alpha e^{-i\tau(1-1/4\bar{n})}\exp\left[\bar{n}\left(e^{-i\tau/2\bar{n}} - 1 + \frac{i\tau}{2\bar{n}}\right)\right] \approx \alpha e^{-i\tau}e^{-T^2/8} \qquad (7.36)$$

where $\tau = \lambda_p g t/2\sqrt{\bar{n}}$ and $T = 2\tau/\sqrt{\bar{n}}$, which means that the phase of the field is approximately equal to $-\tau$, while its amplitude decreases as $e^{-T^2/8}$ (phase diffusion). We also obtain the following approximation for the fluctuations (Equation 7.35):

$$\langle\sigma_q\rangle_p \approx \frac{1}{2} + 2\left[\frac{\bar{n}}{2}\left(e^{-T^2/8} - e^{-T^2/16}\right) + \frac{T^2}{64}\left(e^{-T^2/16} - 4e^{-T^2/8}\right)\right]\cos 2\tau$$
$$- \frac{T\sqrt{\bar{n}}}{4}\left(2e^{-T^2/8} - e^{-T^2/16}\right)\sin 2\tau + \bar{n}\left(1 - e^{-T^2/16}\right) \qquad (7.37)$$

Which, for a very large number of photons, is simplified to

$$\langle\sigma_q\rangle_p \approx \frac{1}{2} + \frac{\tau^2}{4}[1 - \cos 2\tau] + \frac{\tau}{2}\sin 2\tau \tag{7.38}$$

It can be seen from the above expression that the fluctuations are minimized when

$$\tau = k\pi - \frac{1}{2}\mathrm{arccot}\left(\frac{\tau}{2}\right) \approx k\pi - \frac{1}{2}\mathrm{arccot}\left(\frac{k\pi}{2}\right), \quad k = 1, 2, \ldots$$

This means that the maximum squeezing of the field is reached at the times close to the revival times: $t_{\mathrm{sq}} \simeq 0.9T_R, 1.95T_R, \ldots$. Thus, squeezing of the field at long times is generated by each semiclassical eigenstate, due to the deformation of the initial coherent state evolving under the action of a nonlinear effective Hamiltonian (7.21).

The nature of squeezing at short times is different. There is no short-time squeezing if the atomic system is initially prepared in a semiclassical eigenstate, since for short times the deformation generated by the effective Hamiltonian (7.20) is negligible. At short times, field squeezing is due to the interference of the field states corresponding to different semiclassical atomic eigenstates. When an initial coherent state of the field starts to interact with the atomic system, the field quasiprobability distribution Q (which at time $t = 0$ had the form of a single Gaussian hump) separates into various humps, corresponding to contributions of various atomic semiclassical eigenstates into the initial atomic state. For times of the order of the collapse time, $gt \sim 1$, the field coherent states corresponding to different atomic semiclassical states interfere with each other and generate a squeezed field state. It is easy to see from Equation 7.19, that the factorized atomic states are approximately orthogonal for short times and, thus, the factorization approximation (7.17) does not describe the interference between semiclassical eigenstates. Hence, the interference effects are of the order of $A/\sqrt{\bar{n}}$, which is precisely the accuracy of the approximation (7.17) and, thus, the expected squeezing of the field would be also of the order of $A/\sqrt{\bar{n}}$. We find the fluctuations of the field operators by using the evolution operator in the quasi-linear approximation (7.10) in the same way as we did to find $S_z(t)$ in Equation 7.11:

$$\hat{n}(t) = \hat{n} + S_z(1 - \cos\hat{\zeta}) + \frac{1}{2i}\left(S_+ \mathrm{e}^{\mathrm{i}\hat{\phi}} \sin\hat{\zeta} - \sin\hat{\zeta}\mathrm{e}^{-\mathrm{i}\hat{\phi}} S_-\right) \equiv \hat{n} + \hat{L}(\hat{n}) \tag{7.39}$$

$$a(t) = \sqrt{1 + \hat{n}(t)}\exp(\mathrm{i}\hat{\phi}) = \sqrt{1 + \frac{\hat{L}(\hat{n})}{\hat{n}+1}}\,a$$

$$a^2(t) = \sqrt{1 + \frac{\hat{L}(\hat{n})}{\hat{n}+1}}\sqrt{1 + \frac{\hat{L}(\hat{n}+1)}{\hat{n}+2}}\,a^2 \tag{7.40}$$

where $\hat{L}$ is defined by Equation 7.39. Let us calculate squeezing for the completely excited initial atomic state $|A_{\mathrm{at}}$. Using the $SU(2)$ group representation technique,

we obtain

$$
{}_{\mathrm{at}}\langle A|\hat{n}(t)|A\rangle_{\mathrm{at}} = \hat{n} + \frac{A}{2}(1 - \cos\hat{\xi}_A) \tag{7.41}
$$

$$
{}_{\mathrm{at}}\langle A|a(t)|A\rangle_{\mathrm{at}} = \cos^A\left(\frac{\hat{\upsilon}_A - \hat{\upsilon}_{A+1}}{2}\right)\left[1 + \frac{AZ_1}{4(\hat{n}+1)} - \frac{AZ_1(AZ_1 + 1 - Z_1)}{32(\hat{n}+1)^2}\right]a
$$

$$
{}_{\mathrm{at}}\langle A|a^2(t)|A\rangle_{\mathrm{at}} = \cos^A\left(\frac{\hat{\upsilon}_A - \hat{\upsilon}_{A+2}}{2}\right)\left[1 + \frac{AZ_2}{2(\hat{n}+1)} - \frac{AZ_2}{4(\hat{n}+1)^2}\right]a^2 \tag{7.42}
$$

where

$$
Z_j = 1 - \frac{\cos\left[(\hat{\upsilon}_{A+j} + \hat{\upsilon}_A)/2\right]}{\cos\left[(\hat{\upsilon}_{A+j} - \hat{\upsilon}_A)/2\right]}
$$

and

$$
\hat{\upsilon}_{A+j} = 2gt\sqrt{\hat{n} + j + \frac{1}{2} + \frac{A}{2}}
$$

Now we can calculate the fluctuations $\langle\sigma_q\rangle$. Averaging the operators in Equation 7.42 over the initial coherent state (with zero phase) and using

$$
\left\langle\frac{\cos\hat{\upsilon}_A}{\overline{n}+1}\right\rangle \approx \frac{1}{\overline{n}}\left[\langle\cos\hat{\upsilon}_A\rangle + \frac{gt}{\sqrt{\overline{n}_A}}\langle\sin\hat{\upsilon}_A\rangle\right]
$$

$$
\langle\cos\hat{\upsilon}_A\rangle = \exp\left[-\frac{g^2t^2}{2}\frac{\overline{n}}{\overline{n}_A}\right]\cos(2gt\sqrt{\overline{n}_A})
$$

where $\langle\ldots\rangle$ means the average over the coherent state $|\alpha = \sqrt{\overline{n}}\rangle$ and $\overline{n}_A = \overline{n} + 1/2 + A/2$, we obtain

$$
\begin{aligned}
\langle\sigma_q\rangle = {} & \frac{1}{2} + \frac{Agt}{2\sqrt{\overline{n}_A}}\sin(2gt\sqrt{\overline{n}_A})\exp\left(-\frac{(gt)^2}{2}\frac{\overline{n}}{\overline{n}_A}\right) \\
& + \frac{A^2}{16\overline{n}}\left[\exp\left(-2(gt)^2\frac{\overline{n}}{\overline{n}_A}\right) - \exp\left(-\frac{(gt)^2}{2}\frac{\overline{n}}{\overline{n}_A}\right)\right]\cos(4gt\sqrt{\overline{n}_A}) \\
& + \frac{A^2}{16\overline{n}}\left(1 - \mathrm{e}^{-(gt)^2\overline{n}/\overline{n}_A}\right) + O\left(\frac{A^3}{\overline{n}^2}\right)
\end{aligned} \tag{7.43}
$$

From Equation 7.43 we can find the conditions for the best squeezing in terms of the parameters of the system: number of atoms A and $\overline{n}$. The time instant at which the best squeezing occurs can be obtained considering the second term in Equation 7.43, which describes the modulated Rabi oscillations. The local squeezing maxima correspond to the minima of the sine function and occur for times $gt(k) = (k - 1/4)\pi/\sqrt{\overline{n}_A}$, $k = 1, 2, \ldots$. The time t^* of the best squeezing is

$$
t^* = \frac{\pi}{g\sqrt{\overline{n}_A}}\left(\left[\frac{\overline{n}_A}{\pi\sqrt{\overline{n}}}\right] - \frac{1}{4}\right) \tag{7.44}
$$

The value of squeezing depends, of course, on the number of atoms. By evaluating the fluctuations $\langle\sigma_q\rangle$ at time t^*, we find

$$\langle\sigma_q\rangle_{\min} \simeq \frac{1}{2}\left(1 - d\frac{A}{\sqrt{\overline{n}}} + b\frac{A^2}{8\overline{n}}\right) \tag{7.45}$$

where $d = \mathrm{e}^{-1/2} \approx 0.606$ and $b = 1 + d - d^2 - d^4 \approx 1.103$. As expected, squeezing increases with the number of atoms that participate in the interaction. Also, from the equation above the optimal number of atoms for the coherent state with a given average number of photons can be evaluated

$$A_{\mathrm{opt}} \sim 2.4\sqrt{\overline{n}} \tag{7.46}$$

which is consistent with the supposition $A \ll \overline{n}$ if $A/\overline{n} \approx 2.4/\sqrt{\overline{n}} \ll 1$ is satisfied. From Equations 7.44 and 7.46 we can obtain the maximum absolute value of the squeezing, which is of the order of 66%.

7.6
Coherence Transfer Between Atoms and Field

During the evolution process, the atomic system may transfer part of its initial coherence to the field [51, 85]. This is shown, for example, by the fact that the field that was in a phaseless state, that is, whose phase distribution function was simply $P_A(\varphi) = 1/2\pi$, acquires a certain phase during the interaction with the atomic system that has been prepared in a special state. The phase becomes better defined when the size of the atomic system increases [85]. This effect may be quite pronounced even in the case of a strong initial field. In this section, we consider the interaction of a set of A two-level atoms prepared initially in a semiclassical eigenstate (7.13) with the field in the number state, $|N\rangle$, $N \gg A$. For simplicity, we consider the atomic semiclassical eigenstate $|\underline{p = 0}\rangle$,

$$|\mathrm{in}\rangle = |\underline{0}\rangle_{\mathrm{at}}|N\rangle_f \tag{7.47}$$

and find the evolution of the phase operator $\exp(\mathrm{i}k\hat{\phi})$, where k is an integer, for this initial state of the system. By applying the evolution operator in the form (7.10) to the phase operator and by evaluating the averages over the initial state (Equation 7.47), we obtain

$$\langle\mathrm{in}|\exp\left(\mathrm{i}k\hat{\phi}(t)\right)|\mathrm{in}\rangle = \sum_{l=0}^{A} c_{l+k}\, c_l^*\, d^A_{ll+k}(\theta_k)\,, \quad |k| \le A$$

$$\langle\mathrm{in}|\exp\left(\mathrm{i}k\hat{\phi}(t)\right)|\mathrm{in}\rangle = 0, \quad |k| > A$$

where $c_l = \langle l|\underline{0}\rangle$, matrix elements of finite rotations from $(A+1)$-dimensional representation of the $SU(2)$ group, $P^A_{lk}(\theta) = \langle l|\exp(\mathrm{i}\theta S_x)|k\rangle$, are defined in Appendix 11.5, and

$$\theta_k = 2\left(\sqrt{N+l-\frac{A}{2}+\frac{1}{2}} - \sqrt{N+l-\frac{A}{2}+\frac{1}{2}+k}\right) \tag{7.48}$$

Since $|k|, l \leq A$, the argument θ_k can be simplified in the limit of a strong initial field $N \gg A$ in the following way:

$$\theta_k \simeq -2k\theta, \quad \theta = \frac{gt}{2\sqrt{N_A}}, \quad N_A = N - \frac{A}{2} + \frac{1}{2}$$

Taking into account the explicit expression for the coefficients c_l,

$$c_l = \frac{(-1)^l}{2^{A/2}}\sqrt{\frac{A!}{l!(A-l)!}}$$

and the integral representation for the $P^A_{lk}(\theta)$ function (see Appendix 11.5), we obtain

$$\langle \mathrm{in}|\exp(\mathrm{i}k\hat{\phi}(t))|\mathrm{in}\rangle = (-\mathrm{i})^k \frac{A!}{(A+k)!}\, P^k_A(\cos k\theta) \tag{7.49}$$

where $P^k_A(x)$ are associated Legendre polynomials [49]. Using Equation 7.49 we can reconstruct the phase distribution function as

$$\begin{aligned} P_A(\varphi) &= \frac{1}{2\pi}\sum_{k=-A}^{A} \langle \mathrm{in}|\exp(\mathrm{i}k\hat{\phi}(t))|\mathrm{in}\rangle\, \mathrm{e}^{-\mathrm{i}k\varphi} \\ &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{A}(-1)^k \frac{A!}{(A+k)!}\, P^k_A(\cos k\theta)\cos\left(\varphi k - \frac{\pi k}{2}\right) \end{aligned} \tag{7.50}$$

One can see that the mean value of $\cos\hat{\phi}$ is always zero (this implies that the phase is located at $\varphi = 0$), while the average of $\sin\widehat{\phi}$ is of the form

$$\langle \sin\hat{\phi}\rangle = -\frac{1}{A+1}\, P^1_A(\cos\theta)$$

and reaches its maximum value close to the times

$$gt \simeq \frac{\pi\sqrt{N_A}}{A} \tag{7.51}$$

Also, it is easy to obtain from Equation 7.50 the fluctuations of the operators $\cos\hat{\phi}$ and $\sin\hat{\phi}$, which are of the form

$$\sigma_{\cos\phi} = \frac{1}{2} - \frac{1}{2(A+1)(A+2)} P^2_A(\cos 2\theta) \tag{7.52}$$

$$\sigma_{\sin\phi} = \frac{1}{2} + \frac{1}{2(A+1)(A+2)} P^2_A(\cos 2\theta) - \left[\frac{1}{(A+1)} P^1_A(\cos\theta)\right]^2 \tag{7.53}$$

Considering asymptotic relations for the associated Legendre polynomials [49], we find that in the limit of a great number of atoms $A \gg 1$ (but $A \ll N$) the fluctuations (7.52) and (7.53) at times (7.51) tend toward the following values:

$$\sigma_{\cos\phi} = \frac{1}{2}\left(1 - J_2(\pi)\right)$$
$$\sigma_{\sin\phi} = \frac{1}{2} + \frac{1}{2}J_2(\pi) - \left[J_1\left(\frac{\pi}{2}\right)\right]^2$$

In the case of a single atom, $A = 1$, the mean value and the fluctuation of $\sin\hat{\phi}$ have the form

$$\langle \sin\hat{\phi}\rangle = \frac{1}{2}\sin\theta$$
$$\sigma_{\sin\phi} = \frac{1}{2} - \frac{1}{4}\sin^2\theta$$

The average $\langle \sin\hat{\phi}\rangle$ reaches its maximum for $gt = \pi\sqrt{N}/2$. The variance $\sigma^6_{\sin\phi}$ at the same time reaches the value 1/4, which means that there is a significant sharpening of the phase distribution function.

7.7
Resonant Fluorescence Spectrum

In this section, we obtain the resonant fluorescence spectrum for the Dicke model in the limit of the strong field using the methods developed before [86, 87]. The physical spectrum is defined in the standard form:

$$S(\nu, t) \equiv 2\gamma \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \mathrm{e}^{-(\gamma - \mathrm{i}\nu)(t-t_2)} \mathrm{e}^{-(\gamma + \mathrm{i}\nu)(t-t_1)} G(t_1, t_2) \tag{7.54}$$

where γ and ν are the bandwidth and the peak of the transmission frequency, respectively, and $G(t_1, t_2)$ is the two-time correlation function defined by

$$\begin{aligned} G(t_1, t_2) &= \langle S_+(t_2) S_-(t_1)\rangle \\ &= \sum_{n=0}^{\infty} P_{n\,f}\langle n|\,_{\mathrm{at}}\langle m| U^\dagger(t_2) S_+ U(t_2) U^\dagger(t_1) S_- U(t_1)|m\rangle_{\mathrm{at}}|n\rangle_f \end{aligned} \tag{7.55}$$

where P_n is the initial photon distribution of the field and $|m\rangle_{\mathrm{at}}$ is the initial state of the atomic system with m atoms excited. The evolution operator in a strong field case is taken in the quasi-linear approximation (7.10), which is equivalent to the zeroth-order approximation of Chapter 6. Using the identity decomposition in the basis of the zeroth-order eigenvectors (6.23)

$$\sum_{Np} |\underline{N, p}\rangle\langle \underline{N, p}| = 1$$

we obtain

$$\begin{aligned} G(t_1,t_2) = {} & \sum_{N=m}^{\infty} P_{N-m} \sum_{p,q,r}^{A} \alpha_{mp}\alpha^{*}_{mr} \langle \underline{N,p}|S_+|\underline{N-1,q}\rangle\langle \underline{N-1,q}|S_-|\underline{N,r}\rangle \\ & \times \mathrm{e}^{\mathrm{i}(\omega+\omega^{n+m}_{q,p})t_2}\,\mathrm{e}^{-\mathrm{i}(\omega+\omega^{n+m}_{q,r})t_1} \end{aligned} \tag{7.56}$$

where $\alpha_{mp} = \langle N,m|\underline{N,p}\rangle$ are the components of the zeroth-order eigenvectors and $\omega^{n}_{q,p}$ are the frequencies of the zeroth-order spectral components

$$\begin{aligned} & \omega^{n}_{q,p} = \Lambda^{n}_{p} - \Lambda^{n-1}_{q}, \quad \Lambda^{n}_{p} = 2g\left(p - \frac{A}{2}\right)\Omega_n \\ & \Omega_n = \sqrt{n - \frac{A}{2} + \frac{1}{2}} \end{aligned} \tag{7.57}$$

Note that in the case of exact resonance, the components α_{mp} and the matrix elements $\langle \underline{N,p}|S_+|\underline{N-1,q}\rangle$ do not depend on N

$$\begin{aligned} \langle \underline{N-1,q}|S_-|\underline{N,p}\rangle = \langle \underline{q}|S_-|\underline{p}\rangle = {} & \left(\frac{A}{2} - p\right)\delta_{q,p} + \frac{1}{2}\sqrt{(p+1)(A-p)}\,\delta_{q,p+1} \\ & - \frac{1}{2}\sqrt{p(A-p+1)}\,\delta_{q,p-1} \end{aligned}$$

Substituting Equation 7.56 in Equation 7.54 we obtain

$$\begin{aligned} S(\Delta_f,t) = {} & 2\gamma \sum_{N=m}^{\infty} P_{N-m} \sum_{p,q,r=0}^{A} \alpha_{mp}\alpha^{*}_{mr}\,\langle \underline{p}|S_+|\underline{q}\rangle\langle \underline{q}|S_-|\underline{r}\rangle \\ & \times \frac{\exp[\mathrm{i}t(\omega^{N}_{q,r} - \omega^{N}_{q,p})]}{[\gamma - \mathrm{i}(\Delta_f - \omega^{N}_{q,r})][\gamma + \mathrm{i}(\Delta_f - \omega^{N}_{q,p})]} \end{aligned} \tag{7.58}$$

where $\Delta_f = \nu - \omega$ is the difference between the filter and the field frequencies. In this equation, the sum over the total number of excitations $N = n + m$ in the system is substituted in the place of the sum over the initial number of photons n. When $p \neq r$, the emission spectrum is composed of terms that contain the products of two complex Lorentzians centered at $\omega^{N}_{q,r}$ and at $\omega^{N}_{q,p}$. Thus, in the case $p \neq r$, the intensities are small and can be neglected. The stationary spectrum of the system is obtained when $p = r$,

$$S(\Delta_f) = 2\gamma \sum_{N=m}^{\infty} P_{N-m} \sum_{p,q=0}^{A} \frac{|\alpha_{mp}|^2 \left|\langle \underline{q}|S_-|\underline{p}\rangle\right|^2}{\gamma^2 + (\Delta_f - \omega^{N}_{q,p})^2} \tag{7.59}$$

and has a Lorentzian form.

Let us consider the case where the field is prepared initially in its *number state* $|N\rangle_f$, thus, in the sum over N, only one term is considered. Letting $q = p + k$, $k = 0, \pm 1$, we obtain from (Equation 7.57) the following expression for the spectral frequencies:

$$\omega^{N}_{p+k.p} = -k\Omega_N + \frac{2g}{\Omega_N}\left(p + k - \frac{A}{2}\right)$$

In this case, the term $k = 0$ corresponds to the central band and the terms with $k = \pm 1$ correspond to the side bands (right and left, respectively). When $k \geq 2$ the matrix element $\langle \underline{p + k}|S_-|\underline{p}\rangle$ is equal to zero in the zeroth-order approximation. If the other terms in the expansion (7.7) are considered, then we would observe more bands in the spectrum. The separation between the side bands and the central band is $2\Omega_N$. Each band consists of lines that form a fine structure (which is a cooperative effect). The index p numbers the lines within a given band; it ranges from 0 to A ($0 \leq p \leq A$) for the central band, while $0 \leq p \leq A - 1$ for the right band and $1 \leq p \leq A$ for the left band. When A is even, the central band consists of A lines, while for an odd A, the band contains $A + 1$ lines. The side bands have A lines each. In Figure 7.4 we observe a cooperative fine structure of the resonant fluorescence spectrum for the field in a number state. The intensity of the spectral lines is determined by the coefficients

$$\mathrm{A}_{q,p} = |\alpha_{mp}|^2 |\langle \underline{q}|S_-|\underline{p}\rangle|^2 \tag{7.60}$$

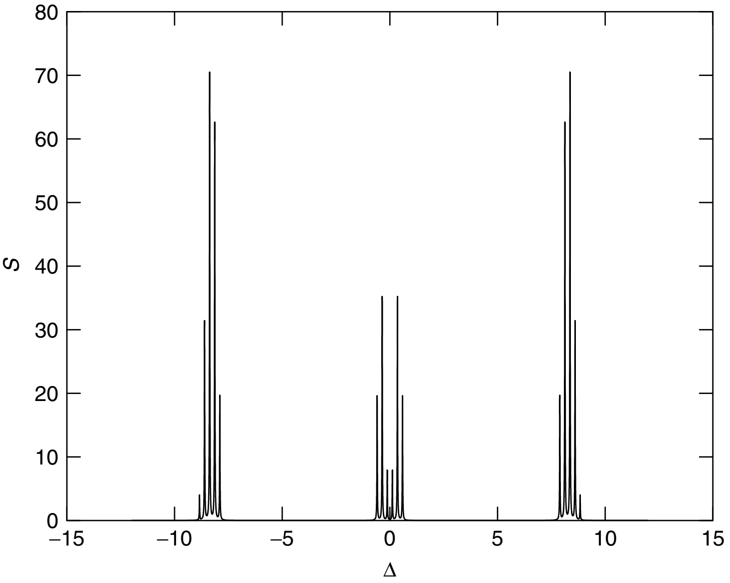

Figure 7.4 Resonance fluorescence spectrum for the field initially in a number state, $n = 20$ and for five initially excited atoms, $\gamma = 0.01$.

When all the atoms are initially in the same state $m = A$ or 0 (excited or not excited), the zeroth-order amplitudes (7.60) are proportional to the binomial coefficients,

$$|\alpha_{0p}|^2 = |\alpha_{Ap}|^2 = \frac{A!}{2^A p!(A-p)!}$$

and the intensities of the peaks that form the basic triplet are

$$\begin{aligned} \mathrm{A}_{p+1,p} &= \frac{A!}{(A-p)!p!}\frac{(p+1)(A-p)}{2^{A+2}} \\ \mathrm{A}_{p,p} &= \frac{A!}{(A-p)!p!}\frac{(A-2p)^2}{2^{A+2}} \\ \mathrm{A}_{p-1,p} &= \frac{A!}{(A-p)!p!}\frac{p(A-p+1)}{2^{A+2}} \end{aligned}$$

for the right, central, and left bands respectively.

7.8
Atomic Systems with *n* Energy Levels

In this section, we show that all the techniques applied in the case of the Dicke model are also valid for more complex atomic systems [88a, 88b]. Let us consider a set of A (identical) atoms with n energy levels interacting with a mode of the quantum field (under the rotating wave approximation) inside a perfect cavity [44]. The Hamiltonian that describes this system has a generic form

$$H = \omega\hat{N} + V, \quad \hat{N} = \hat{n} + h, \quad V = \Delta h_0 + g(aX_+ + a^\dagger X_-) \tag{7.61}$$

where $a, a^\dagger$ and $\hat{n} = a^\dagger a$ are field operators and g is the coupling constant. The free Hamiltonian of the atomic system has been separated into two parts: $h + \Delta h_0$. The diagonal operator h only has *resonant* transitions with the field frequency ω and determines the configuration of the atomic levels, while the second diagonal term Δh_0 determines the shifts from the exact resonances (different atomic transitions may have different shifts). Clearly, $[h, \Delta h_0] = 0$. The atomic operators $X_\pm$ describe transitions between the resonant levels and have the following commutation relations:

$$[h, X_-] = -X_-, \quad [h, X_+] = X_+ \tag{7.62}$$

The Hamiltonian (7.61) and the commutation relations (7.62) describe a system that is quite general; for example, the Dicke model corresponds to a special case of a set of A identical two-level atoms (initially prepared in a symmetrized state) when the atomic operators

$$h = S_z, \quad X_\pm = S_\pm$$

form an $A+1$ dimension representation of the $su(2)$ algebra and satisfy an additional commutation relation: $[S_+, S_-] = 2S_z$. In this case, Δ denotes a real frequency shift, that is, $\Delta h_0 = (\omega_0 - \omega)\, S_z$, where ω_0 is the atomic transition frequency.

In general, the atomic operators (7.62) generate a finite dimension representation of the $su(n)$ algebra [19] so that the free atomic Hamiltonian $h + \Delta h_0$ is expressed in terms of elements of the Cartan subalgebra and the atomic transition operators $X_\pm$ correspond to the simple roots. For the case of identical atoms (which we consider here), the algebra rank depends on the number of levels, while the representation is determined by the number of atoms and the symmetry properties of the initial state of the atomic system under permutations of atoms. We define the basis in the representation space of the atomic algebra as

$$|k,\gamma\rangle_{\mathrm{at}}, \quad 0 \le k \le \mathcal{A}, \quad h|k,\gamma\rangle_{\mathrm{at}} = (k - C)|k,\gamma\rangle_{\mathrm{at}} \tag{7.63}$$

where k is the number of excitations in the atomic system, $\mathcal{A}$ is their maximum value, C is a constant that determines the lowest atomic energy level ($k=0$) andγ denotes all the other atomic indexes (these are atomic level populations). Note that the operator $h + C$ has an integer spectrum. In the case of the $su(2)$ algebra, we have only one diagonal atomic operator: $h = S_z$ and, thus, there are no additional indexes. In other words, for atoms with two levels, atomic inversion completely determines the state of the atomic system (symmetrized with respect to permutations). In this case, the maximum possible number of atomic excitations $\mathcal{A}$ is equal to the number of atoms A, and the choice of the constant $C = A/2$ corresponds to the zero energy level in the middle of the spectrum.

From the commutation relations (7.62), it follows that the operator of the number of excitations commutes with the Hamiltonian $[\hat{N}, H] = 0$. Thus, if the initial state of the system belonged to a subspace with a given number of excitations N, then the system always evolves within this subspace.

We introduce the basis in the space of the total system as

$$|N,k\rangle = |N-k\rangle_f \otimes |k\rangle_{\mathrm{at}}, \quad 0 \le k \le \mathcal{A}, \quad \hat{N}|N,k\rangle = (N - C)|N,k\rangle \tag{7.64}$$

where $|n\rangle_f$ are number states of the field and $|k\rangle_{\mathrm{at}}$ is the atomic basis (7.63) (which we will call the "bare basis"). For a fixed N, the Hamiltonian (7.61) in the basis (7.64) takes on the form of a finite dimension matrix. Its rows and columns are numbered by the indexes k and γ (hereafter, we omit the atomic index γ).

According to the method proposed at the beginning of this chapter, we need to find the evolution operator of the system for the case of a strong initial field, which implies that the average number of photons in the field is much larger than the maximum possible number of excitations in the atomic system: $\overline{n} \gg \mathcal{A}$. We also require that the width of the photon distribution is not very large, or more precisely, that $\Delta n \ll \overline{n}$. We introduce the operator $\hat{Q}$ as follows:

$$\hat{Q} = \exp\left[\mathrm{i}\hat{\phi}(h + C)\right] \tag{7.65}$$

where $\hat{\phi}$ is the phase operator (3.126), (3.127). The operator (7.65) is unitary on the states with $n > \mathcal{A} \ge k$ and its action on the basis elements (7.64) would be

$$\hat{Q}|k\rangle_{\text{at}} \otimes |n\rangle_f = \exp(\mathrm{i}\hat{\phi}k)|k\rangle_{\text{at}} \otimes |n\rangle_f = |k\rangle_{\text{at}} \otimes |n-k\rangle_f = |N=n,k\rangle \tag{7.66}$$

Since the operator $h + C$ has an integer spectrum, the operator $\hat{Q}$ is a direct sum of powers of the phase operator. The operator (7.65) obeys the following commutation relations:

$$\hat{Q}f(\hat{n})\hat{Q}^{\dagger} = f(\hat{n} + h + C) \tag{7.67}$$

$$\hat{Q}X_{+}\hat{Q}^{\dagger} = \exp(\mathrm{i}\hat{\phi})X_{+}, \qquad \hat{Q}X_{-}\hat{Q}^{\dagger} = \exp(-\mathrm{i}\hat{\phi})X_{-} \tag{7.68}$$

Applying the transformation (7.65) to the Hamiltonian (7.61) and using the relations (7.67), (7.68) we obtain

$$\hat{H} = \hat{Q}H_{\text{at}}(\hat{n})\hat{Q}^{\dagger} \tag{7.69}$$

$$H_{\text{at}}(\hat{\nu}) \equiv \hat{\nu}\omega + \Delta h_0 + g\left(\sqrt{\hat{\nu} - h + 1}\, X_{+} + X_{-}\sqrt{\hat{\nu} - h + 1}\right) \tag{7.70}$$

where $\hat{\nu} = \hat{n} - C$. The Hamiltonian (7.70) is diagonal in the field space. Using

$$\hat{Q}^{\dagger}|m\rangle_{\text{at}} = |m\rangle_{\text{at}} \exp(-\mathrm{i}\hat{\phi}m), \quad {}_{\text{at}}\langle k|\hat{Q} = \exp(\mathrm{i}\hat{\phi}k)\,{}_{\text{at}}\langle k|,$$

we obtain for the evolution operator of the system:

$$U = \exp(-\mathrm{i}tH) = \hat{Q}\exp\left[-\mathrm{i}tH_{\text{at}}(\hat{\nu})\right]\hat{Q}^{\dagger} \tag{7.71}$$

The expression above reduces the problem of finding the evolution operator for the whole system to finding the atomic evolution operator $U_{\text{at}}(\nu) = \exp\left[-\mathrm{i}tH_{\text{at}}(\hat{\nu})\right]$ (we note that the operator $\hat{\nu}$ commutes with all the other operators in $H_{\text{at}}(\hat{\nu})$ and can be treated as a c- number). Matrix elements of the evolution operator (7.71) in the atomic basis (being operators in the field space) take on the form

$${}_{\text{at}}\langle k|U|m\rangle_{\text{at}} = \mathrm{e}^{\mathrm{i}\hat{\phi}k}\,{}_{\text{at}}\langle k|\exp\left[-\mathrm{i}tH_{\text{at}}(\hat{\nu})\right]|m\rangle_{\text{at}}\,\mathrm{e}^{-\mathrm{i}\hat{\phi}m} \tag{7.72}$$

$$= {}_{\text{at}}\langle k|\exp\left[-\mathrm{i}tH_{\text{at}}(k + \hat{\nu})\right]|m\rangle_{\text{at}}\mathrm{e}^{\mathrm{i}\hat{\phi}(k-m)} \tag{7.73}$$

$$= \mathrm{e}^{\mathrm{i}\hat{\phi}(k-m)}\,{}_{\text{at}}\langle k|\exp\left[-\mathrm{i}tH_{\text{at}}(m + \hat{\nu})\right]|m\rangle_{\text{at}} \tag{7.74}$$

The Equations 7.72 to 7.74 are exact under the condition that the operator $\hat{Q}$ is unitary on the initial state, that is that $n > \mathcal{A}$.

In the case of a strong initial field we expand the roots in (Equation 7.70) powers of a small parameter $(\hat{\nu} + 1/2)^{-1} \ll 1$:

$$H_{\text{at}} = H_0(\hat{\nu}) + \frac{1}{\sqrt{\hat{\nu} + \frac{1}{2}}}H_1 + O(\nu^{-3/2}) \tag{7.75}$$

$$H_0 = \omega\widehat{\nu} + \Delta h_0 + g\sqrt{\hat{\nu} + \frac{1}{2}}(X_{+} + X_{-}) \tag{7.76}$$

$$H_1 = -\frac{g}{4}\{h,\, X_{+} + X_{-}\}_{+} \tag{7.77}$$

or, in general,

$$H_{\mathrm{at}} = \omega\hat{\nu} + \Delta h_0 + g\sum_{k=0}^{\infty} \frac{c_k}{2^k\left(\hat{\nu}+1/2\right)^{k-1/2}} \left\{h,\ldots\left\{h, X_+ + X_-\right\}_+\right\}_+$$

where the coefficients c_k are coefficients in the expansion

$$\sqrt{1-x} = \sum_{k=0}^{\infty} c_k x^k$$

and $\{,\}_+$ is the anticommutator. From Equation 7.76 we see that the zeroth-order Hamiltonian H_0 is linear in generators of the atomic algebra and, thus, can be easily diagonalized using the standard methods of the $su(n)$ algebra representation theory. In this way, the total system evolution operator (7.71) in the zeroth-order approximation is constructed from the evolution operator for the atomic system in the classical field

$$\begin{aligned} U_{\mathrm{cl}}(\nu) &= \exp[-\mathrm{i}tH_{\mathrm{cl}}(\nu)] \\ H_{\mathrm{cl}} &= \Delta h_0 + g\sqrt{\varepsilon}\,(X_+ + X_-) \end{aligned} \tag{7.78}$$

by substituting the amplitude of the classical field $\sqrt{\varepsilon}$ by the operator $\sqrt{\hat{\nu}+1/2}$:

$$U \simeq \hat{Q}\exp\left[-\mathrm{i}t\left(\omega\hat{\nu} + H_{\mathrm{cl}}\left(\sqrt{\varepsilon} \to \sqrt{\hat{\nu}+\frac{1}{2}}\right)\right)\right]\hat{Q}^{\dagger} \tag{7.79}$$

Now it is easy to prove that the factorization of the wave function is a general property that all the atomic systems described by a Hamiltonian of the type (7.61) have. We introduce atomic semiclassical eigenstates as the eigenstates of the operator $X_+ + X_-$ (which is the Hamiltonian of the atomic system in a classical field):

$$(X_+ + X_-)\,|\underline{p}\rangle = \lambda_p|\underline{p}\rangle \tag{7.80}$$

Applying the zeroth-order evolution operator (7.79) to the initial state (for simplicity we take the initial coherent state with zero phase),

$$|\mathrm{in}\rangle = |\underline{p}\rangle_{\mathrm{at}} \otimes |\alpha\rangle_f, \quad p = 1,\ldots,\mathcal{A} \tag{7.81}$$

and using the same approximations as in section 5.2, we obtain the wave function, which is approximately factorized into the atomic and field parts:

$$\begin{aligned} |\Psi(t)\rangle &= U|\mathrm{in}\rangle \cong |\Phi_p(t)\rangle_f \otimes |A_p(t)\rangle_{\mathrm{at}} \\ |\Phi_p(t)\rangle_f &= \exp\left[-\mathrm{i}gt\lambda_p\sqrt{\hat{\nu}+\frac{1}{2}}\right]|\alpha\rangle_f \\ |A_p(t)\rangle_{\mathrm{at}} &= \exp\left[-\mathrm{i}\omega_p t\left(h+C\right)\right]|\underline{p}\rangle_{\mathrm{at}} \end{aligned} \tag{7.82}$$

where

$$\omega_p = \frac{g\lambda_p}{2\sqrt{\overline{n} - C + 1/2}}$$

The factorization (7.82) is valid up to the times $gt \sim \overline{n}$ with an error $\sim O(\overline{n}^{-1/2})$ in the coefficients.

The physical consequences of the factorization (7.82) are basically the same as in the Dicke model. For example, it can easily be seen that the states (7.81) are trapped states, since the average of operator h (which has a sense of the atomic inversion operator) does not evolve in time if the system is prepared initially in one of its semiclassical eigenstates (7.81):

$$\langle \text{in}|U^{+}(t)hU(t)|\text{in}\rangle = \langle A_p(t)|h|A_p(t)\rangle = {}_{\text{at}}\langle \underline{p}|h|\underline{p}\rangle_{\text{at}} = \text{const}$$

Also, from the factorization (7.82), it follows that the collapses and revivals are common effects that would be observed in any atomic system (which was initially prepared in a superposition of its semiclassical eigenstates (7.80)) interacting with a strong coherent quantum field. Indeed, an arbitrary atomic state can be expanded over the semiclassical eigenstates (7.80) as

$$|\psi(0)\rangle_{\text{at}} = \sum_p c_p|\underline{p}\rangle_{\text{at}} \tag{7.83}$$

Thus, for the atomic inversion we have

$${}_f\langle\alpha|\,{}_{\text{at}}\langle\psi(0)|U^{+}hU|\psi(0)\rangle_{\text{at}}|\alpha\rangle_f = \sum_{p,q} c_q^{*}c_p\langle\Phi_q(t)|\Phi_p(t)\rangle\langle A_q(t)|h|A_p(t)\rangle$$

The matrix element of the field has the same form as in Equation 7.27 and generates a revival – collapse structure with collapse times

$$T_C \sim \frac{2}{g\left(\lambda_p - \lambda_q\right)}\sqrt{\frac{\overline{n} - C + 1/2}{\overline{n}}}$$

and possible revival times

$$T_R = \frac{2\pi k}{\omega_p - \omega_q}, \quad k = 1, 2, \ldots \tag{7.84}$$

The amplitude of the revivals is determined by the atomic matrix elements

$$\langle A_q(t)|h|A_p(t)\rangle \sim {}_{\text{at}}\langle \underline{q}|he^{\mathrm{i}t(\omega_q - \omega_p)(h+C)}|\underline{p}\rangle_{\text{at}}$$

Since at possible revival times (Equation 7.84), we have

$$\exp\left[2\pi\mathrm{i}\left(h + C\right)\right] = 1$$

the atomic matrix element becomes,

$$\langle A_q(t)|h|A_p(t)\rangle \sim {}_{\text{at}}\langle \underline{q}|h|\underline{p}\ \rangle_{\text{at}} \tag{7.85}$$

The calculation of this matrix element involves the knowledge of the atomic algebra representation. In the case of the Dicke model, ${}_{\rm at}\langle \underline{q}|h|\underline{p}\,\rangle_{\rm at} \sim \delta_{p\,q\pm 1}$ and, as we know already, only one series of collapses and revivals survives.

As an example more complicated than the Dicke model, we consider the atomic systems that consist of A identical atoms with three energy levels (with different configurations) interacting with a mode of the quantum field.

Let us use the Schwinger realization (see Appendix 11.1) to construct collective atomic operators (1.49):

$$S^{ij} = b_j^\dagger b_i, \quad i,j = 1,2,3, \quad S^{11} + S^{22} + S^{33} = A$$

$$\left[b_i, b_j^\dagger\right] = \delta_{ij}, \quad \left[b_i, b_j\right] = \left[b_i^\dagger, b_j^\dagger\right] = 0$$

which act in the $(A+1)(A+2)/2$ dimension space formed by the vectors $|n_1, n_2, n_3\rangle$, $n_1 + n_2 + n_3 = A$, where n_i is the number of atoms with the ith energy level excited. In other words, the atomic inversion operators S_z^{ij}

$$S_z^{ij} = \frac{1}{2}\left(S^{jj} - S^{ii}\right)$$

and the transition operators $S_\pm^{ij}$

$$S_+^{ij} = S^{ij} = \left(S_-^{ij}\right)^\dagger, \quad S_-^{ij} = S^{ji}, \quad i,j = 1,2,3, \quad i < j$$

constitute a symmetric $(A+1)(A+2)/2$ dimension representation of the $su(3)$ algebra.

The interaction Hamiltonians that correspond to different atomic configurations in terms of collective operators have the following forms.

7.8.1
Cascade Configuration Ξ

$$H_\Xi = \omega a^\dagger a + H_{0,\rm at} + g_1\left(aS_+^{12} + a^\dagger S_-^{12}\right) + g_2\left(aS_+^{23} + a^\dagger S_-^{23}\right) \tag{7.86}$$

where $H_{0,\rm at}$ is the free atomic Hamiltonian $H_{0,at} = \sum_{i=1}^{3} E_i S^{ii}$, and E_i is the energy of the ith atomic level. The Hamiltonian Equation 7.86 has the following integral of motion:

$$N_\Xi = a^\dagger a + 2\left(S_z^{12} + S_z^{23}\right) = a^\dagger a + 2S_z^{13}, \quad [H, N_\Xi] = 0 \tag{7.87}$$

It is clear that N_Ξ has an integer spectrum. Thus, the Hamiltonian (7.86) can be rewritten as

$$H_\Xi = \omega N + \frac{2}{3}\left(2\Delta_2 + \Delta_1\right)S_z^{23} + \frac{2}{3}\left(2\Delta_1 + \Delta_2\right)S_z^{12} + EA + H_{\rm int} \tag{7.88}$$

$$H_{\rm int} = g_2\left(aS_+^{23} + a^\dagger S_-^{23}\right) + g_1\left(aS_+^{12} + a^\dagger S_-^{12}\right)$$

where we introduce the shifts

$$\Delta_{21} = E_2 - E_1 - \omega, \quad \Delta_{32} = E_3 - E_2 - \omega \tag{7.89}$$

and $E = (E_0 + E_1 + E_2)/3$ is the average energy. By defining the operators $X_\pm$, h, Δh as

$$X_\pm = g_2 S_\pm^{23} + g_1 S_\pm^{12}, \quad h_\Xi = 2\left(S_z^{12} + S_z^{23}\right) = 2S_z^{13}$$

$$\Delta h = \frac{2}{3}\left(2\Delta_{32} + \Delta_{21}\right) S_z^{23} + \frac{2}{3}\left(2\Delta_{21} + \Delta_{32}\right) S_z^{12} + EA$$

the Hamiltonian (7.88) takes on the form (7.61). The operator $\hat{Q}$, which diagonalizes the Hamiltonian (7.88) is of the form

$$\hat{Q} = \exp\left[2\mathrm{i}\hat{\phi}\left(S_z^{12} + S_z^{23}\right)\right]$$

It can easily be seen that the relation (7.70) takes on the form,

$$H_{\text{at}}(\hat{n}) = \hat{Q}^\dagger H_\Xi \hat{Q} = \omega\hat{n} + \Delta h + EA + g_2\left(\sqrt{\hat{n} - h_\Xi + 1}\, S_+^{23} + h.c.\right)$$
$$+ g_1\left(\sqrt{\hat{n} - h_\Xi + 1}\, S_+^{12} + h.c.\right)$$

7.8.2
Λ-Type Configuration

$$H_\Lambda = \omega a^\dagger a + H_{0,\text{at}} + g_1\left(aS_+^{13} + a^\dagger S_-^{13}\right) + g_2\left(aS_+^{23} + a^\dagger S_-^{23}\right) \tag{7.90}$$

The Hamiltonian Equation 7.90 has the integral of motion

$$N_\Lambda = a^\dagger a + h_\Lambda, \quad h_\Lambda = \frac{2}{3}\left(S_z^{13} + S_z^{23}\right), \quad [H, N_\Lambda] = 0$$

and its spectrum is not an integer since

$$h_\Lambda |n_1, n_2, n_3\rangle = (n_3 - C)\,|n_1, n_2, n_3\rangle$$

where $C = A/3$. The Hamiltonian (7.90) can be written in the form (7.61) with $X_\pm$, h, Δh defined as

$$X_\pm = g_2 S_\pm^{13} + g_1 S_\pm^{23}, \quad h_\Lambda = \frac{2}{3}\left(S_z^{13} + S_z^{23}\right)$$

$$\Delta h = \frac{2}{3}\left(2\Delta_{31} - \Delta_{32}\right) S_z^{13} + \frac{2}{3}\left(2\Delta_{32} - \Delta_{31}\right) S_z^{23} + EA$$

where the shifts are defined by Equation 7.89 and

$$\Delta_{31} = E_3 - E_1 - \omega \tag{7.91}$$

The operator $\hat{Q}$ that diagonalizes the Hamiltonian (7.90) is of the form

$$\hat{Q} = \exp\left[\frac{2}{3}\mathrm{i}\hat{\phi}\left(S_z^{13} + S_z^{23} + \frac{A}{3}\right)\right]$$

7.8.3
V-Type Configuration

$$H_V = \omega a^\dagger a + H_{0,\text{at}} + g_1\left(aS_+^{13} + a^\dagger S_-^{13}\right) + g_2\left(aS_+^{12} + a^\dagger S_-^{12}\right) \tag{7.92}$$

The Hamiltonian (7.92) has the integral of motion

$$N_V = a^\dagger a + h_V, \quad h_V = \frac{2}{3}\left(S_z^{13} + S_z^{12}\right), \quad [H, N_V] = 0$$

and its spectrum is not an integer either, since

$$h_V|n_1, n_2, n_3\rangle = (n_3 + n_2 - C)\,|n_1, n_2, n_3\rangle$$

where $C = 2A/3$. The Hamiltonian (7.92) can be written in the form (7.61) with $X_\pm$, h, Δh defined as

$$X_\pm = g_2 S_\pm^{13} + g_1 S_\pm^{12}, \quad h = \frac{2}{3}\left(S_z^{13} + S_z^{12}\right)$$

$$\Delta h = \frac{2}{3}\left(2\Delta_{21} - \Delta_{31}\right) S_z^{12} + \frac{2}{3}\left(2\Delta_{31} - \Delta_{21}\right) S_z^{13} + EA$$

where the shifts are defined by Equations 7.89 and 7.91. The operator $\hat{Q}$ that diagonalizes the Hamiltonian (7.92) has the form

$$\hat{Q} = \exp\left[\frac{2}{3}\mathrm{i}\hat{\phi}\left(S_z^{13} + S_z^{12} + \frac{2A}{3}\right)\right]$$

7.9
Dicke Model in the Dispersive Limit

In this section, we describe the case where the shift between the field frequency and the atomic transition frequency is large compared to the frequency of Rabi oscillations $gA\sqrt{(\bar{n}+1)} \ll \Delta$ (the dispersive limit of the Dicke model [89, 89c]). In fact, in this section, we do not consider the strong field limit. The Hamiltonian (7.5) in the interaction representation takes on the form

$$H = \Delta S_z + g\left(aS_+ + a^\dagger S_-\right) \tag{7.93}$$

Clearly, in the dispersive limit, the field does not cause transitions between the atomic levels (at least, not in the first-order approximation), but only causes phase shifts for these levels. Thus, the effective Hamiltonian that describes the dispersive dynamics should be diagonal in the atomic variables. To find this effective Hamiltonian, we will follow Lie transformation method (Appendix 11.4) and apply a unitary transformation

$$U = \exp\left[\frac{g}{\Delta}\left(aS_+ - a^\dagger S_-\right)\right] \tag{7.94}$$

to the Hamiltonian (7.93):

$$H_1 = UHU^\dagger \tag{7.95}$$

Keeping only the leading order terms on $g/\Delta \ll 1$, we obtain from the Hamiltonian (7.95), the effective Hamiltonian

$$H_1 \approx \Delta S_z + \eta\left(2\hat{n}+1\right) S_z + \eta\left(C_2 - S_z^2\right) \tag{7.96}$$

where $\eta = g^2/\Delta,\ C_2 = (A/2+1)\,A/2$ is the Casimir operator of the $su(2)$ algebra, and $\hat{n}$ is the photon number operator. It can easily be seen that the dynamics of any observable is not significantly affected by the transformation U, since this transformation is time-independent and only introduces small corrections to the operator amplitudes in the Heisenberg representation. The time range in which the effective Hamiltonian (7.96) describes the evolution of the system well (i.e. when the neglected terms do not affect dynamics) can also be estimated, giving $At\left(\sqrt{\overline{n}}g\right)^3/\Delta^2 \ll 1$.

In the case of one atom, the effective Hamiltonian (7.96) coincides with that found for the JCM (Equation 5.11). It is important to note that the effective Hamiltonian (7.96) for the case of many atoms differs essentially from the one atom case due to the presence of the nonlinear term S_z^2, which can lead to correlations among the atoms and thus produce the atomic squeezed states.

As an example, we find the evolution of the field and atomic density matrices for an arbitrary initial atomic state

$$|\mathrm{in}\rangle_{\mathrm{at}} = \sum_{p=0}^{A} c_p |p\rangle_{\mathrm{at}} \tag{7.97}$$

and the initial coherent state of the field $|in\rangle_f = |\alpha\rangle_f$. In the course of the dispersive evolution, the field density matrix takes on the form of an incoherent superposition of coherent states

$$\rho_{\mathrm{f}}(t) = \sum_{p=0}^{A} |c_p|^2 |\alpha \mathrm{e}^{-\mathrm{i}\varphi_p t}\rangle_{\mathrm{f}\,\mathrm{f}}\langle \alpha \mathrm{e}^{-\mathrm{i}\varphi_p t}|$$

where $\varphi_p = \omega_f + \eta\left(2p - A\right)$ and the atomic density matrix is

$$\begin{aligned}\rho_{\mathrm{at}}(t) = \sum_{p,q=0}^{A} & c_p c_q^* |p\rangle_{\mathrm{at\,at}}\langle q| \exp\left[-\overline{n}\left(1-\exp\left[2\mathrm{i}t\eta\left(q-p\right)\right]\right)\right] \\ & \times \exp\left[\mathrm{i}t\left(\eta\left(q+p+1\right)+\mathrm{i}t\omega_a\right)\left(q-p\right)\right]\end{aligned} \tag{7.98}$$

where $\overline{n} = |\alpha|^2$. From Equation 7.98, one can observe that the initial atomic state is reconstructed at times $t = m\pi/\eta$, $m = 1, 2, \ldots$. At these times, the field and the atomic system are uncoupled, and the total system is in its factorized state.

Since the Hamiltonian (7.96) is diagonal both in the atomic space and in the field space, the average number of photons and atomic excitations are conserved. However, the nondiagonal operators evolve with time. For example, for the dipole momentum operator $S_x(t)$ in the Heisenberg representation, we obtain

$$S_x(t) = \mathrm{e}^{\mathrm{i}tH_1} S_x \mathrm{e}^{-\mathrm{i}tH_2} = S_+ \exp\left[\mathrm{i}t\left(\Delta + 2\eta\left(\hat{n} - S_z\right)\right)\right] + h.c. \tag{7.99}$$

It is obvious that for a bare initial atomic state (an eigenstate of the operator S_z) and an arbitrary state of the field, the average of the dipole momentum is zero: $\langle S_x(t)\rangle = 0$. However, for a coherent initial atomic state $|\xi\rangle_{\mathrm{at}}$ and for the field in the number state, we have

$$\langle S_x\rangle = A_\xi \,\mathrm{Re}\left[\xi^* \mathrm{e}^{i\vartheta(t)+2\mathrm{i}tN\eta}\left(1+|\xi|^2\exp\left[-2\mathrm{i}t\eta\right]\right)^{A-1}\right]$$

where

$$A_\xi = \frac{A}{\left(1+|\xi|^2\right)^A}, \quad \vartheta(t) = t\left(\Delta + A\eta\right)$$

If the field is initially in the coherent state $|\alpha\rangle$, we obtain

$$\langle S_x\rangle = A_\xi \,\mathrm{Re}\left[\xi^* \mathrm{e}^{\mathrm{i}\vartheta(t)}\exp\left[-\overline{n}\left(1-\exp\left[2\mathrm{i}t\eta\right]\right)\right]\left(1+|\xi|^2\exp\left[-2\mathrm{i}t\eta\right]\right)^{A-1}\right]$$

In the last case, if the field is strong, $\overline{n} \gg 1$, structures similar to collapses and revivals can be observed. For the case of a single atom, $A = 1$, a regular structure appears. For the case of many atoms, this structure is modulated by a collective factor $\left(1+|\xi|^2\exp\left[-2\mathrm{i}t\eta\right]\right)^{A-1}$, which is the consequence of the nonlinearity of the Hamiltonian (7.96).

Being nonlinear, the Hamiltonian (7.96) can generate atomic squeezed states from the coherent atomic states. As has been mentioned already, the evolution operator (1.26)

$$U(t) = \exp\left[-\mathrm{i}t\eta S_z^2\right] \tag{7.100}$$

rotates the operators $S_\pm$ in a nonlinear way (that is, the rotation parameter is proportional to the operator S_z), which leads to twisting of quantum fluctuations. Thus, significant squeezing of the atomic states can be achieved. However, due to the dependence of the Hamiltonian (7.96) on the photon number operator, the evolution of the atomic fluctuations is different for different initial states of the field. It is obvious that squeezed atomic states are generated if the field initially is in a number state. Indeed, in this case, the evolution operator differs from Equation 7.100 only by a global phase

$$\mathrm{e}^{-\mathrm{i}t[\Delta+\eta(1+2N)]S_z} \tag{7.101}$$

which rotates the initial atomic state around the z axis:

$$\mathrm{e}^{-\mathrm{i}\gamma S_z}|\xi\rangle_{\mathrm{at}} = \mathrm{e}^{\mathrm{i}\gamma A/2}|\xi \mathrm{e}^{-\mathrm{i}\gamma}\rangle_{\mathrm{at}} \tag{7.102}$$

However, for any other initial state of the field

$$|\varphi\rangle_f = \sum_n \beta_n |n\rangle_f$$

the phase in Equation 7.102 must be considered as an operator:

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}t\left[\Delta+\eta\left(1+2\hat{n}\right)\right]S_z}|\xi\rangle_{\mathrm{at}}|\varphi\rangle_f &= \mathrm{e}^{\mathrm{i}t(\Delta+\eta)A/2}\\ &\quad\times\sum_n \beta_n \mathrm{e}^{\mathrm{i}t\eta nA}|\xi \mathrm{e}^{-\mathrm{i}t[\Delta+\eta(1+2n)]}\rangle_{\mathrm{at}}|n\rangle_f\end{aligned} \tag{7.103}$$

As it can be shown, this operator not only rotates the coherent atomic state but also generates superpositions of these states with different phases. Thus, though the nonlinear part of the Hamiltonian (7.96) twists each coherent atomic state in the superposition (Equation 7.103), the degree of atomic squeezing may decrease drastically.

7.10
Two-Photon Dicke Model

Now we consider a set of "cascade" configuration three-level atoms, such that the intermediate state $|2\rangle_{(j)}$, with energy E_2 for each atom, is coupled via allowed dipole transitions to the states $|1\rangle_{(j)}$ with energy E_1 and $|3\rangle_{(j)}$ with energy E_3, where $E_1 < E_2 < E_3$, interacting with a mode of the electromagnetic field inside a perfect cavity. The field frequency is ω and the two-photon resonance condition is satisfied,

$$E_3 - E_2 = \omega + \Delta, \quad E_2 - E_1 = \omega - \Delta, \quad E_3 - E_1 = 2\omega \tag{7.104}$$

where we have introduced the shift Δ of the intermediate energy level from the resonance with the field frequency. Using the integral of motion (7.87), the Hamiltonian (7.86) can be rewritten as follows:

$$\begin{aligned} H &= \omega N + H_I + EA \\ H_I &= \frac{2\Delta}{3}\left(S_z^{23} - S_z^{12}\right) + g_1\left(aS_+^{12} + a^\dagger S_-^{12}\right) + g_2\left(aS_+^{23} + a^\dagger S_-^{23}\right) \end{aligned} \tag{7.105}$$

Now, we assume that the shift Δ of the intermediate atomic energy level of the field frequency is very large. Obviously, this energy level practically does not participate in the exchange of populations between levels $|1\rangle$ and $|3\rangle$. However, the presence of this level may introduce a phase shift for the levels $|1\rangle$ and $|3\rangle$. To find the effective Hamiltonian that describes the dynamics of the levels $|1\rangle$ and $|3\rangle$), we transform the Hamiltonian (7.105) by a sequence of small rotations [90] according to the general method of the Lie-type transformations (see Appendix 11.4):

$$\begin{aligned} U_1 &= \exp\left[\frac{g_1}{\Delta}\left(aS_+^{12} - a^\dagger S_-^{12}\right)\right] \\ U_2 &= \exp\left[\frac{g_2}{\Delta}\left(aS_+^{23} - a^\dagger S_-^{23}\right)\right] \end{aligned} \tag{7.106}$$

We assume that the exponents in the formulae above are small, which means that $\Delta \gg Ag_1\sqrt{\overline{n}}, Ag_2\sqrt{\overline{n}}$ (here $\overline{n}$ is the average number of photons in the field). The Hamiltonian (7.105) is transformed in the following way:

$$H_2 = U_2 U_1 H U_1^\dagger U_2^\dagger \tag{7.107}$$

Using the formula (11.33) and keeping the terms up to the first order in $g_{1,2}/\Delta \ll 1$, we obtain

$$\begin{aligned} H_2 \approx {} & \frac{2\Delta}{3}\left(S_z^{23} - S_z^{12}\right) + \frac{g_1 g_2}{\Delta}\left(a^2 S_+^{13} + a^{\dagger 2} S_-^{13}\right) \\ & - \frac{g_1^2}{\Delta} P_{12} + \frac{g_2^2}{\Delta} P_{23} - \frac{g_1 g_2}{\Delta}\left(S_+^{12} S_-^{23} + S_-^{12} S_+^{23}\right) \end{aligned} \tag{7.108}$$

where

$$P_{12} = \left[a S_+^{12}, a^\dagger S_-^{12}\right] = S_+^{12} S_-^{12} + 2a^\dagger a S_z^{12}$$
$$P_{23} = \left[a S_+^{23}, a^\dagger S_-^{23}\right] = S_+^{23} S_-^{23} + 2a^\dagger a S_z^{23}$$

and we have neglected the terms of order $\left(g\sqrt{\bar{n}}/\Delta\right)^2$. It can easily be seen that the last term in Equation 7.108 can be eliminated by the transformation

$$U_3 = \exp\left[\frac{g_1 g_2}{2\Delta^2}\left(S_+^{12} S_-^{23} - S_-^{12} S_+^{23}\right)\right]$$

which does not introduce corrections to the terms of order $g_{1,2}/\Delta$. Using the atomic population operators $S^{jj}, j = 1, 2, 3$ we obtain

$$P_{12} = S^{22}\left(S^{11} + a^\dagger a + 1\right) - a^\dagger a S^{11}$$
$$P_{23} = S^{22}\left(S^{11} - a^\dagger a\right) + S^{33}\left(a^\dagger a + 1\right)$$

The Hamiltonian (7.108) takes the following form:

$$\begin{aligned} H_2 \approx {} & \frac{\Delta}{3}\left(S^{11} + S^{33}\right) + \frac{g_1 g_2}{\Delta}\left(a^2 S_+^{13} + a^{\dagger 2} S_-^{13}\right) + \frac{g_1^2}{\Delta} a^\dagger a S^{11} + \frac{g_2^2}{\Delta} S^{33}(a^\dagger a + 1) \\ & - S^{22}\left[\frac{2\Delta}{3} + \frac{g_1^2}{\Delta}\left(S^{11} + a^\dagger a + 1\right) - \frac{g_2^2}{\Delta}\left(S^{11} - a^\dagger a\right)\right] \end{aligned} \tag{7.109}$$

Since the operator S^{22} is a constant of motion for the Hamiltonian H_2, the last term in the Hamiltonian (7.109) can be omitted under the condition that initially the energy level $|2\rangle$ is not populated. In this case, $S^{11} + S^{33} = A$ and introducing

$$S_z^{13} = \frac{1}{2}\left(S^{33} - S^{11}\right)$$

we rewrite the Hamiltonian (7.109) in its final form as

$$H_{\text{eff}} = A\beta_1 \hat{n} + \left[(\beta_2 - \beta_1)\hat{n} + \beta_2\right]\left(S_z^{13} + \frac{A}{2}\right) + \lambda\left(a^2 S_+^{13} + a^{\dagger 2} S_-^{13}\right) \tag{7.110}$$

where $\hat{n} = a^\dagger a$ is the operator of the number of photons.

$$\beta_1 = \frac{g_1^2}{\Delta}, \quad \beta_2 = \frac{g_2^2}{\Delta}, \quad \lambda = \frac{g_1 g_2}{\Delta}$$

are the new interaction constants omitting the constant $A\Delta/3$. The Hamiltonian (7.110) describes the effective interaction of a set of two-level atoms with a quantized field mode under two-photon resonance condition. The second term represents the so-called dynamic Stark shift [62] and is a trace left by level $|2\rangle$ in the effective Hamiltonian. Due to the approximations made, the effective Hamiltonian describes well the dynamics of the system for the times $gt \ll \Delta^2/g^2\overline{n}$, where $\overline{n}$ is the average number of photons in the field.

Since only the atomic transitions between levels 1 and 3 survive, henceforth we omit the atomic indexes and use the notation $|k, A\rangle$ for the atomic basis where the operator $S_z \equiv S_z^{13}$ is diagonal

$$S_z|k, A\rangle = \left(k - \frac{A}{2}\right)|k, A\rangle, \quad k = 0, \dots, A$$

The Hamiltonian (7.110) is nonlinear and cannot be solved exactly for an arbitrary number of atoms. However, in the case of a strong quantum field, $\overline{n} \gg A$, we can develop a perturbation theory similar to the one applied in the case of the one-photon Dicke model. In analogy to Equation (7.1), we introduce the transformation $\hat{Q}$ as follows:

$$\hat{Q} = \exp\left[2\mathrm{i}\hat{\phi}\left(S_z + \frac{A}{2}\right)\right]$$

This transformation allows us to diagonalize the Hamiltonian (7.110) in the field space:

$$H_{\mathrm{at}}\left(\hat{n}\right) = \hat{Q}^{\dagger} H_{\mathrm{eff}} \hat{Q}$$

which yields

$$\begin{aligned} H_{\mathrm{at}}\left(\hat{n}\right) = {} & \frac{A}{2}\left[(\beta_2 + \beta_1)\left(\hat{n} - 2S_z - A\right) + \beta_2\right] \\ & + \left[(\beta_2 - \beta_1)\left(\hat{n} - 2S_z - A\right) + \beta_2\right] S_z + g(\hat{n}, S_z)S_+ + S_- g(\hat{n}, S_z) \end{aligned}$$

where

$$g(\hat{n}, S_z) = \lambda\sqrt{(\hat{n} - 2S_z - A + 1)(\hat{n} - 2S_z - A + 2)}$$

In a strong field limit, we can expand the roots in the equation above over the powers of the small parameter $A/\hat{n}$:

$$H_{\mathrm{at}}\left(\hat{n}\right) = H_0 + H_1 + O\left(\nu^{-1}\right)$$

where

$$\begin{aligned} H_0 = {} & \frac{A}{2}\left[(\beta_2 + \beta_1)\hat{\nu} + \beta_2\right] + 2\lambda\left[\hat{\nu} + \frac{1}{2}\right] S_x \\ & + \left[(\beta_2 - \beta_1)\hat{\nu} + \beta_2 - (\beta_2 + \beta_1) A\right] S_z \\ H_1 = {} & -2\lambda\left\{S_z, S_x\right\} - 2\left(\beta_2 - \beta_1\right) S_z^2 \end{aligned}$$

Here, $\hat{\nu} = \hat{n} - A$. The Hamiltonian H_0 is linear in the generators of the $su\,(2)$ algebra and is proportional to the large parameter $\hat{\nu}$ (this choice of the parameter was taken in order to eliminate the first-order corrections to the eigenvalues in the case $\beta_1 = \beta_2$). The Hamiltonian H_1 is nonlinear and independent of $\hat{\nu}$. Corrections to the eigenvalues of H_0 generated by H_1 are of the order A^2 and cannot be ignored. (Let us recall that in the one-photon Dicke model, the nonlinear part is nondiagonal and is of order $\nu^{-1/2}$ and, thus, can be neglected in the first approximation. That is, the corrections generated by this nonlinear part tend to zero when $\overline{n} \to \infty$). To consider all the significant terms, we divide the Hamiltonian $H_0 + H_1$ into a diagonal part (which includes the complete H_0 and a part of H_1) and in a nondiagonal part (which comes solely from H_1). Thus, the first-order correction generated by the nondiagonal part, becomes zero and the second-order correction is proportional to ν^{-1}. With an appropriate choice of the basis, the diagonal part of $H_0 + H_1$ is of the form

$$H_{\mathrm{d}} = \frac{A}{2}\left[(\beta_2 + \beta_1)\,\hat{\nu} + \beta_2\right] + \mu\left(aS_x + bS_z\right) + c\left(aS_x + bS_z\right)^2 \tag{7.111}$$

where

$$\begin{gathered}
\mu\left(\hat{\nu}\right) = \sqrt{4\lambda^2\left(\hat{\nu} + \frac{1}{2}\right)^2 + \left((\beta_2 - \beta_1)\,\hat{\nu} + \beta_2 - (\beta_2 + \beta_1)\,A\right)^2} \\
a(\hat{\nu}) = \frac{2\lambda}{\mu}\left(\hat{\nu} + \frac{1}{2}\right), \quad b(\hat{\nu}) = \frac{(\beta_2 - \beta_1)\,\hat{\nu} + \beta_2 - (\beta_2 + \beta_1)\,A}{\mu} \\
a^2 + b^2 = 1, \quad c = -2\,(\beta_2 - \beta_1)\left(b^2 - \frac{a^2}{2}\right) - 6\lambda ab
\end{gathered} \tag{7.112}$$

The Hamiltonian (7.111) is nonlinear in the generators of the $su(2)$ algebra. However, the contribution of the nonlinear part is important (i.e. it does not disappear with the increase of the photon number $\overline{n} \to \infty$) only if $\beta_1 \neq \beta_2$. The last term in Equation 7.111 represents the shift in the field frequency. Let us note also that the Hamiltonian (7.111) is essentially different from the one obtained in the same approximation for the common one-photon Dicke model, where the Hamiltonian is a linear form in the generators of the $su\,(2)$ algebra. The contribution of the nondiagonal part to the eigenvalues of the Hamiltonian H_{d} is of order ν^{-1}. Finally, the evolution operator takes on the form

$$U(t) = \hat{Q}\,U_d(t)\,\hat{Q}^{\dagger} \tag{7.113}$$

where

$$U_d(t) = \exp\left[-\mathrm{i}tH_{\mathrm{d}}\right]$$

We can also rewrite the Hamiltonian (7.111) in the following way:

$$H_d = \mathrm{e}^{-\mathrm{i}\theta S_y}\left(\mu\left(\hat{\nu}\right) S_z + cS_z^2 + \frac{A}{2}\left[(\beta_2 + \beta_1)\,\hat{\nu} + \beta_2\right]\right)\mathrm{e}^{\mathrm{i}\theta S_y} \tag{7.114}$$

where $\theta = \arctan a/b$ and in the limit of the strong field

$$\begin{aligned} &\mu\left(\hat{\upsilon}\right) \approx \hat{\upsilon}\chi + \varphi, && c \approx -2\left(\beta_2 - \beta_1\right), \\ &\chi = \beta_2 + \beta_1, && \varphi = \beta_2 - (\beta_2 - \beta_1)A \end{aligned} \tag{7.115}$$

The nonlinear term $\sim S_z^2$ indicates that atomic squeezing (in the collection of effective two-level systems with transitions between the levels 1 and 3) can be generated in the process of evolution if the condition $\beta_2 \neq \beta_1$ is fulfilled. This is an important difference from the standard Dicke model, when the nonlinear corrections to the effective Hamiltonian $\sim S_x$, see Equations 7.8 and 7.9, tend to zero with increase in the field intensity, so that in the strong field limit, no atomic squeezing can be generated.

The matrix elements of the evolution operator in the atomic basis are

$$\begin{aligned} \langle k|\,U(t)|m\rangle = \mathrm{e}^{2\mathrm{i}\hat{\phi}m} \sum_{l=0}^{A} d_{ml}(-\theta) d_{lk}(\theta) \mathrm{e}^{-\mathrm{i}t\left[(l-A/2)\mu\left(\hat{\upsilon}\right)+(l-A/2)^2 c\right]} \\ \times\, \mathrm{e}^{-2\mathrm{i}\hat{\phi}k} \mathrm{e}^{-\mathrm{i}t(A/2)\left[(\beta_2+\beta_1)\left(\hat{\upsilon}+2k\right)+\beta_2\right]} \end{aligned} \tag{7.116}$$

where $d_{lk}(\theta)$ is the Wigner function:

$$d_{lk}(\theta) = \langle k|\mathrm{e}^{i\theta S_y}|m\rangle$$

In the case of one atom, $A = 1$, the matrix (7.116) is of the form

$$U(t) = \begin{bmatrix} \dfrac{\left(\beta_2(\hat{n}+1)\mathrm{e}^{-\mathrm{i}tM} + \beta_1(\hat{n}+2)\right)}{M}, & \dfrac{\lambda a^2\left(\mathrm{e}^{-\mathrm{i}tN}-1\right)}{N} \\ \dfrac{\lambda a^{\dagger 2}\left(\mathrm{e}^{-\mathrm{i}tM}-1\right)}{M}, & \dfrac{\left(\beta_1 \hat{n}\mathrm{e}^{-\mathrm{i}tN}+\beta_2(\hat{n}-1)\right)}{N} \end{bmatrix} \tag{7.117}$$

where $N = \beta_2\left(\hat{n}-1\right) + \beta_1\hat{n}$, $M = \beta_2\left(\hat{n}+1\right) + \beta_1\left(\hat{n}+2\right)$

Using the evolution operator in the form (7.113), we can find the operator S_z in the Heisenberg representation:

$$\begin{aligned} S_z(t) &= U^{\dagger}(t) S_z U(t) = \hat{Q}^{\dagger}\widetilde{S}_z(t)\hat{Q} \\ \widetilde{S}_z(t) &= b\left(aS_x + bS_z\right) - \frac{a}{2}\left\{\left(bS_x - iS_y - aS_z\right)\mathrm{e}^{-\mathrm{i}t\left(\mu-c+2c\left(aS_x+bS_z\right)\right)} + h.c.\right\} \end{aligned}$$

or in the normally ordered form

$$\begin{aligned} \widetilde{S}_z(t) = b\left(aS_x + bS_z\right) \\ -\frac{a}{2}\left\{\left(\frac{b-1}{2}S_+ - aS_z + \frac{b+1}{2}\left(DS_+ - BS_z\right)\right)\mathrm{e}^{-\mathrm{i}t\left(\mu-c+2c\left(aS_x+bS_z\right)\right)} \right. \\ \left. + \frac{b+1}{2}\mathrm{e}^{-\mathrm{i}t\left(\mu-c+2c\left(aS_x+bS_z\right)\right)} CS_- + h.c.\right\} \end{aligned} \tag{7.118}$$

where

$$D = \frac{a^2\left(\cos 2ct - \mathrm{i}b^3 \sin 2ct - 1\right)}{2\left(\cos ct + \mathrm{i}b \sin ct\right)^2}, \qquad B = -\frac{2a\mathrm{i} \sin ct}{\cos ct + \mathrm{i}b \sin ct}$$

$$C = \frac{1}{\left(\cos ct + \mathrm{i}b \sin ct\right)^2}$$

Note that the inverse transformations for Equation 7.107 can be neglected when calculating the evolution of any observable, since these transformations are independent of time and would only introduce corrections of order $1/\Delta$ in the amplitudes.

As in the case of the one-photon Dicke model, the evolution of some special states in the highly excited coherent field $|\alpha\rangle_f$, $\alpha = |\alpha|\mathrm{e}^{i\varphi}$, $|\alpha| = \sqrt{\bar{n}} \gg 1$, leads to the factorization of the wave function. Let us consider the evolution of the following initial atomic state:

$$|\mathrm{in}\rangle_{\mathrm{at}} = |\underline{p, \varphi}\rangle_{\mathrm{at}} = \mathrm{e}^{-\mathrm{i}\varphi S_z}|\underline{p}\rangle_{\mathrm{at}}, \quad p = 0, 1, \ldots, A \tag{7.119}$$

such that

$$|\underline{p}\rangle_{\mathrm{at}} = \mathrm{e}^{-\mathrm{i}\bar{\theta} S_y}|p\rangle_{\mathrm{at}}, \quad S_z|p\rangle = \lambda_p|p\rangle$$

$$\bar{\theta} = \arctan\frac{\bar{a}}{\bar{b}}, \quad \lambda_p = \left(p - \frac{A}{2}\right)$$

where φ is the phase of the coherent state. It can easily be seen that $|\underline{p}\rangle_{\mathrm{at}}$ is the eigenstate of the operator $\left(\bar{a}S_x + \bar{b}S_z\right)$:

$$\left(\bar{a}S_x + \bar{b}S_z\right)|\underline{p}\rangle_{\mathrm{at}} = \left(p - \frac{A}{2}\right)|\underline{p}\rangle_{\mathrm{at}}$$

where $\bar{\nu} = \bar{n} - A$ and we use the notation $\bar{a} \equiv a(\bar{\nu})$ and $\bar{b} \equiv b(\bar{\nu})$ according to Equation 7.112. For a strong coherent field state, the substitution $a(n), b(n)$ for $\bar{a}, \bar{b}$ leads to an error of order $1/\bar{n}$ in the matrix elements of the evolution operator (7.113). Applying the evolution operator (7.113) to the initial state $|\underline{p, \varphi}\rangle_{\mathrm{at}} \otimes |\alpha\rangle_f$, and making the same approximations as in Section 2 we obtain

$$|\psi(t)\rangle \approx \mathrm{e}^{\mathrm{i}2/3\Delta At}|A_p(t)\rangle_{\mathrm{at}}|\Phi_p(t)\rangle_f \tag{7.120}$$

$$|A_p(t)\rangle_{\mathrm{at}} = \exp\left\{-\mathrm{i}t\left(\omega_p S_z + c\lambda_p^2\right)\right\}|\mathrm{in}\rangle_{\mathrm{at}} \tag{7.121}$$

$$|\Phi_p(t)\rangle_f = \exp\left\{-\mathrm{i}t\left[\omega_p\hat{\nu} + \varphi\lambda_p + \frac{A}{2}\beta_2\right]\right\}|\alpha\rangle_f \tag{7.122}$$

where $\omega_p = p\left(\beta_2 + \beta_1\right)$. The factorization (7.120) is valid for the times $\lambda t \ll \bar{n}$ with the accuracy $O\left(A/\sqrt{\bar{n}}\right)$ in the coefficients. We can see from Equation 7.122 that the wave function of the field can be represented as

$$|\Phi_p(t)\rangle_f = \exp\left(-\mathrm{i}t\psi_p\right)\exp\left[-\mathrm{i}t\omega_p\hat{n}\right]|\alpha_p\rangle_f = \exp\left(-\mathrm{i}t\psi_p\right)|\alpha_p(t)\rangle_f \tag{7.123}$$

where

$$\alpha_p(t) = \alpha\exp\left[-\mathrm{i}t\omega_p\right], \quad \psi_p = \left[\beta_2 - 2A\left(\beta_2-\beta_1\right)\right]p + \left(\beta_2-\beta_1\right)\frac{A^2}{2}$$

From Equation 7.123, it follows that for the initial atomic states (7.119), the evolution of the field is described by the following effective Hamiltonian:

$$H_f = p\left(\beta_2+\beta_1\right)\hat{n}$$

This Hamiltonian produces a rotation around the origin in the phase space of the field. In the phase space, each factorized state can be described by its function Q, which has the form of a hump rotating around a circle of radius $\sqrt{\bar{n}}$ with an angular velocity ω_p and without phase dispersion. This situation differs drastically from the evolution in the single-photon Dicke model, when the Hamiltonian (7.20) that describes the evolution of the field is not linear ($\sim\sqrt{\bar{n}}$) and the hump that corresponds to each factorized state undergoes a phase diffusion (Equation 7.21). The evolution of an arbitrary atomic state

$$|\psi\rangle_{\mathrm{at}} = \sum_p c_p|\underline{p}\rangle_{\mathrm{at}}$$

in the strong coherent field is represented as a superposition of factorized states

$$|\Psi(t)\rangle = \sum_{p=0}^{A} c_p|A_p(t)\rangle_{at}|\Phi_p(t)\rangle_f \tag{7.124}$$

Using Equation 7.124, we calculate the evolution of the atomic inversion as

$$\langle\Psi(t)|S_z|\Psi(t)\rangle = \sum_{p,q} c_p c_q^{*}\,{}_{\mathrm{at}}\langle A_q(t)|S_z|A_p(t)\rangle_{\mathrm{at}}\,{}_f\langle\Phi_q(t)|\Phi_p(t)\rangle_f$$

Using the approximation (7.123), we find, for the matrix element of the field,

$${}_f\langle\Phi_q(t)|\Phi_p(t)\rangle_f = \exp\left[-\mathrm{i}t\left(\psi_p-\psi_q\right)\right]\langle\alpha_q(t)|\alpha_p(t)\rangle$$

which due to

$$|\langle\alpha_q(t)|\alpha_p(t)\rangle| = \exp\left[-\bar{n}\sin^2\frac{\left(\beta_2+\beta_1\right)\left(p-q\right)t}{2}\right] \tag{7.125}$$

differs essentially from zero only in the vicinity of the times

$$T \sim \frac{2\pi k}{\left(\beta_2+\beta_1\right)\left(p-q\right)}, \quad k = 0, 1, \ldots \tag{7.126}$$

The width of this vicinity is of the order $1/\sqrt{\bar{n}}$. The atomic matrix element at the times close to Equation 7.126 takes on the form

$$\begin{aligned}{}_{\mathrm{at}}\langle A_q(t)|S_z|A_p(t)\rangle_{\mathrm{at}} &= \exp\left\{-\mathrm{i}tc\left(\lambda_p^2-\lambda_q^2\right)\right\}\langle q|\mathrm{e}^{-\mathrm{i}\theta S_y}S_z\mathrm{e}^{\mathrm{i}\theta S_y}|p\rangle\\ &= \exp\left\{-\mathrm{i}tc\left(\lambda_p^2-\lambda_q^2\right)\right\}\langle q|aS_x+bS_z|p\rangle\end{aligned} \tag{7.127}$$

Since $\langle q|S_x|p\rangle \sim \delta_{p,q\pm 1}$, we see that there is the only one revival–collapse series (as in the one-photon Dicke model) with the revival times

$$T_R = \frac{2\pi k}{(\beta_2+\beta_1)}, \quad k = 0, 1, \ldots$$

The oscillation period within each revival is

$$T_0 = \frac{2\pi}{\overline{n}\,(\beta_2+\beta_1)}$$

From Equation 7.125 we easily find that the collapse time is of order

$$T_C = \frac{2}{\sqrt{\overline{n}}\,(\beta_2+\beta_1)}$$

Note that the revival times do not depend on the initial intensity of the field, while the collapse time is inversely proportional to the square root of the average number of photons. Due to the factor $\exp\left\{-\mathrm{i}tc\left(\lambda_p^2-\lambda_q^2\right)\right\}$ in Equation 7.127 the atomic matrix element, in contrast with the usual Dicke model, depends on the index p. This modulates the system of collapses and revivals. The modulation period is of the order of the revival time and proportional to the difference $\beta_2-\beta_1$ (in the case in which $\beta_2=\beta_1$ the modulation is still present, but its period increases). This modulation is a collective effect and disappears in the case of one atom.

As in the one-photon Dicke model, the collapses and revivals can be explained in terms of the motion of the humps of the functions Q that correspond to the factorized states in the phase space of the field. A generic atomic state causes the appearance of $A+1$ humps, which rotate with angular velocities $\omega_p = p\,(\beta_2+\beta_1)$. When all the humps are well separated, there are no oscillations of the atomic inversion (collapse region). The collision of two humps with nearby angular velocities ω_p and ω_{p+1} yields a revival of the oscillations. The collision of two humps with ω_p and ω_{p+k}, $k \geq 2$ does not produce a revival. It is interesting to note that all the humps move in the same direction (in contrast with the one-photon Dicke model). Due to the absence of phase dispersion, the humps conserve their shape for a long time. This leads to a very regular collapse and revival structure. Thus, there is no strong squeezing of the field at the revival times, in contrast with what occurs in the one-photon Dicke model, where the phase diffusion produces field squeezing at revival times. On the other hand, squeezing of the field can be expected for short times, close to the time of the collapse, which would essentially be of the same nature as in the case of the one-photon Dicke model. Also, due to the regularity of the system, an amount of squeezing would also be produced at times close to the revival times and would have the same origin as squeezing for the short times.

Using Equation 7.118 one can calculate the evolution of the atomic inversion in a more precise way. For example, taking the field to be initially in a coherent state with zero phase and the atomic system to be in its lowest state, $|\mathrm{in}\rangle = |\alpha\rangle_f|0\rangle_{\mathrm{at}}$, and using $\langle 0|\exp\left[\mathrm{i}\tau\left(aS_x+bS_z\right)\right]|0\rangle = [\cos\tau - \mathrm{i}b\sin\tau]^A$, we have,

$$\langle \mathrm{in}|S_z(t)|\mathrm{in}\rangle \approx -\frac{A\overline{b}^2}{2} - \frac{A\overline{a}^2}{2}\,\mathrm{Re}\left[\left(\cos ct + \mathrm{i}\overline{b}\sin ct\right)^{A-1}\langle \mathrm{e}^{-\mathrm{i}t\mu}\rangle\right] \qquad (7.128)$$

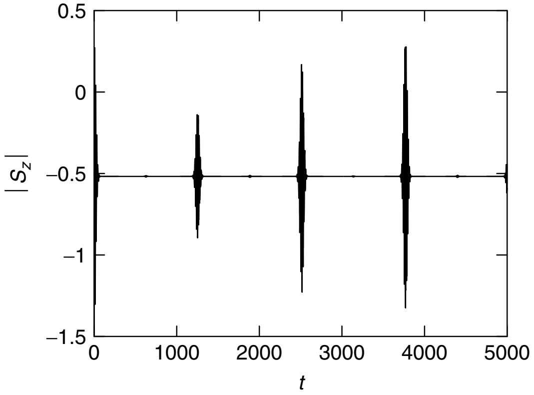

Figure 7.5 Atomic inversion evolution in the case of two-photon resonance: modulated structure of collapses and revivals; field initially in its coherent state, $\overline{n} = 100$ and atoms, $A = 3$, initially unexcited; $g_1 = 1$, $g_2 = 2$, $\Delta = 1000$.

where $\langle \ldots \rangle$ means the average over the Poisson distribution:

$$\langle \mathrm{e}^{-\mathrm{i}t\mu} \rangle = \mathrm{e}^{-\mathrm{i}t(\varphi - \chi A)} \sum_n P_n \mathrm{e}^{-\mathrm{i}tn\chi} = \mathrm{e}^{-\mathrm{i}t(\varphi - \chi A/2)} \exp\left[-\overline{n}\left(1 - \mathrm{e}^{-\mathrm{i}t\chi}\right)\right]$$

where the constants χ, φ, c are defined in Equation 7.115. The substitution of a, b for $\overline{a}, \overline{b}$ leads to an error in the coefficients of the order of $O(1/\overline{n})$. Note that the modulation factors, which are present in Equation 7.128 (see also Figure 7.5) have a collective nature and disappear in the case of one atom, as we have seen already. If $A = 1$, we obtain from Equation 7.128

$$\begin{aligned}\langle \mathrm{in}|S_z(t)|\mathrm{in}\rangle &= -\frac{\overline{b}^2}{2} - \frac{\overline{a}^2}{2}\langle \cos t\mu \rangle \\ &= -\frac{\overline{b}^2}{2} - \frac{\overline{a}^2}{2} \exp\left[-2\overline{n}\sin^2\frac{t\chi}{2}\right] \cos\left(\varphi(t)\right)\end{aligned}$$

where $\varphi(t) = \overline{n}\sin t\chi + (\varphi - \chi A)\,t$, which yields a regular unmodulated structure of collapses and revivals.

7.11
Effective Transitions in Three-Level Atoms with Λ Configuration

Now let us consider a Λ configuration of levels $|1\rangle$, $|2\rangle$, and $|3\rangle$, with the energies $E_1 < E_2 < E_3$, allowed (dipole) transitions $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$, and the forbidden transition $1 \leftrightarrow 2$. The Hamiltonian that describes the evolution of A identical three-level atoms with Λ configuration interacting with a single quantum field mode has a form

$$H_\Lambda = \omega_f a^\dagger a + H_0^{(\mathrm{at})} + H_{\mathrm{int}}, \qquad (7.129)$$
$$H_{\mathrm{int}} = g_{13}(aS_+^{13} + a^\dagger S_-^{13}) + g_{23}(aS_+^{23} + a^\dagger S_-^{23})$$

where $H_0^{(\mathrm{at})}$ is of the form

$$H_0^{(at)} = E_1 S^{11} + E_2 S^{22} + E_3 S^{33}$$

where S^{ii} is the operator of the population of the ith energy level and $S_\pm^{ij}$ $(i < j)$ describes the transitions between levels i and j. Operators $(S^{11}, S^{33}, S_\pm^{13})$ and $(S^{22}, S^{33}, S_\pm^{23})$ form the two $u(2)$ subalgebras that describe the transitions $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$, and satisfy the commutation relations

$$[S^{11}, S_\pm^{13}] = \mp S_\pm^{13}, \quad [S^{33}, S_\pm^{13}] = \mp S_\pm^{13} \qquad (7.130)$$
$$[S_+^{13}, S_-^{13}] = S^{33} - S^{11} = 2S_z^{13} \qquad (7.131)$$
$$[S^{22}, S_\pm^{23}] = \mp S_\pm^{23}, \quad S^{33}, S_\pm^{23}] = \mp S_\pm^{23} \qquad (7.132)$$
$$[S_+^{23}, S_-^{23}] = S^{33} - S^{22} = 2S_z^{23} \qquad (7.133)$$

Since the transitions $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$ are not physically independent, one needs also the following commutation relations:

$$[S_+^{13}, S_+^{23}] = -S_+^{12}, \quad [S_-^{12}, S_-^{23}] = S_-^{12}, \quad [S_+^{13}, S_-^{23}] = 0 \qquad (7.134)$$

where the operators $S_\pm^{12}$ represent the transitions between levels 1 and 2 (these transitions do not appear in the Hamiltonian due to dipole selection rules). The Hamiltonian (7.129) admits the integral of motion

$$N_\Lambda = a^\dagger a + S^{33}, \quad [N_\Lambda, H_\Lambda] = 0 \qquad (7.135)$$

and can be rewritten (up to constant terms) in the following way:

$$H = \omega_f N_\Lambda + H_{\mathrm{int}}$$
$$H_{\mathrm{int}} = -\Delta_{31}X^{11} - \Delta_{32}X^{22} \qquad (7.136)$$
$$+ g_{13}(X_+^{13} + X_-^{13}) + g_{23}(X_+^{23} + X_-^{23})$$

where

$$\Delta_{31} = E_3 - E_1 - \omega_f, \quad \Delta_{32} = E_3 - E_2 - \omega_f$$

and the deformed operators of the $u(3)$ algebra are defined as

$$X_+^{13} = aS_+^{13}, \quad X_z^{13} = S_z^{13}, \quad X_+^{23} = aS_+^{23}, \quad X_z^{23} = S_z^{23} \qquad (7.137)$$

The deformed generators (7.137) satisfy a set of commutation relations similar to the relations (7.130) and (7.132), but instead of the relations (7.131), (7.133), and (7.134), the following commutators should be introduced:

$$[X_+^{13}, X_-^{23}] = \left(S^{33} - a^\dagger a\right) S_+^{12} \equiv Y_+^{12} \tag{7.138}$$
$$[X_+^{12}, X_-^{23}] = -\left(S^{33} - a^\dagger a\right) S_+^{12} \equiv -Y_-^{12}$$

such that

$$[Y_+^{12}, Y_-^{12}] = Q_3(X_z^{12}, X_z^{23}, N_\Lambda)$$

is a third-order polynomial with a complicated expression of very little use for our purposes.

We focus on the dispersive regime in which

$$|\Delta_{13}| \gg Ag_{13}\sqrt{\overline{n}+1}, \quad |\Delta_{23}| \gg Ag_{23}\sqrt{\overline{n}+1} \tag{7.139}$$

where $\overline{n}$ is the average number of photons in the field; i.e. the field is far from resonance with the transitions $1 \to 3$ and $2 \to 3$. Applying the following small rotation operators (see Appendix 11.4)

$$U_1 = \exp[\varepsilon_1(X_+^{13} - X_-^{13})], \quad \varepsilon_1 = \frac{g_{13}}{\Delta_{31}} \ll 1$$

and

$$U_2 = \exp[\varepsilon_2(X_+^{23} - X_-^{23})], \quad \varepsilon_2 = \frac{g_{23}}{\Delta_{32}} \ll 1$$

to the interaction Hamiltonian (7.136), we eliminate the terms that represent transitions $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$, obtaining

$$\begin{aligned} H_{\text{eff}} = U_2 U_1 H_{\text{int}} U_1^\dagger U_2^\dagger &\simeq -(\Delta_{31}S^{11} + \Delta_{32}S^{22}) \\ &+ \varepsilon_1 g_{13}\left[\left(S^{11}+1\right)S^{33} + a^\dagger a(S^{33} - S^{11})\right] \\ &+ \varepsilon_2 g_{23}\left[\left(S^{22}+1\right)S^{33} + a^\dagger a(S^{33} - S^{22})\right] \\ &+ \varepsilon_1 g_{23}(S_+^{12} + S_-^{12})\left(S^{33} - a^\dagger a\right) \end{aligned} \tag{7.140}$$

It is easy to see that both operators $a^\dagger a$ and S^{33} are integrals of motion, so that the effective Hamiltonian (7.140) is basically of the form (2.20). The first term corresponds to the free atomic dynamics. The two second terms represent the dynamic Stark shift. The last term describes an effective interaction between levels 1 and 2 . The important point is that it corresponds to a population transfer (and not just a phase transfer, as could be expected of a dispersive interaction) between these two levels without exchanging photons. The intensity of the transition $1 \leftrightarrow 2$ depends on the difference between the population of level 3 and the number of photons. In other words, there is no population transfer between levels 1 and 2 when the number of photons is exactly equal to the initial population of level 3. It can easily be seen that the effective transitions $1 \leftrightarrow 2$ are strong in the case in which $\Delta_{31} = \Delta_{32}$, i.e. levels 1 and 2 have the same energy (Zeeman type systems).

7.12
N-Level Atoms of Cascade Configuration

Finally, we consider a specific case of N-level atomic systems where some general considerations about effective transitions under different resonant conditions can be made: the cascade configuration. The Hamiltonian governing the evolution of a collection of A identical N-level atoms of a cascade configuration, such that $E_i < E_j$ for $i < j$, interacting with a quantum field has the form

$$H = \sum_{j=1}^{N} E_j S^{jj} + \omega_f a^{\dagger} a + \sum_{j=1}^{N-1} g_j (a S_{+}^{jj+1} + a^{\dagger} S_{-}^{jj+1}) \tag{7.141}$$

where S^{ii} $(i = 1, \ldots, N)$ are the population operators of the ith energy level, and $S_{+}^{ij} = S^{ij}$, $S_{-}^{ij} = \left(S_{+}^{ij}\right)^{\dagger}$ $(i < j)$ describe transitions between levels i and j. The operators S^{ij} form $u(N)$ algebra and satisfy the commutation relations $\left[S^{ij}, S^{km}\right] = \delta_{im} S^{jk} - \delta_{jk} S^{im}$. Due to the integral of motion

$$\sum_{j=1}^{N} S^{jj} = A \tag{7.142}$$

one can introduce population inversion-like operators S_z^{jj+1}

$$S_z^{jj+1} = \frac{1}{2}\left(S^{j+1j+1} - S^{jj}\right), \quad 1 \leq j \leq N-1$$

Note, that the operators $(S_z^{jj+1}, S_{\pm}^{ij})$ form an $su(N)$ algebra. Apart from Equation 7.142 the Hamiltonian (7.141) possesses the following integral of motion:

$$\hat{N} = a^{\dagger} a + \sum_{j=1}^{N-1} \mu_j S_z^{jj+1} \tag{7.143}$$

with $\mu_j = j(N-j)$. Let us introduce detunings between energy levels of the atomic system according to

$$\Delta_j = E_j - E_1 - (j-1)\,\omega_f, \quad \Delta_1 = 0$$

and suppose that Δ_j satisfy the following resonant condition:

$$\Delta_N = 0 \tag{7.144}$$

This means that the field is in an $(N-1)$-photon resonance with the atomic system: $E_N - E_1 = (N-1)\omega_f$. Then the Hamiltonian (7.141) takes the following form:

$$H = \omega_f \hat{N} + EA + H_{\text{int}}, \quad E = \frac{1}{2}(E_N + E_1)$$

$$H_{\text{int}} = h_0 + V,$$

$$h_0 = \sum_{j=1}^{N} \Delta_j S^{jj}, \quad V = \sum_{j=1}^{N-1} g_j (a S_{+}^{jj+1} + a^{\dagger} S_{-}^{jj+1})$$

According to the general scheme (see Appendix 11.4), we introduce the transformation

$$U_1 = \exp\left[T_1\right], \quad T_1 = \sum_{j=1}^{N-1} \alpha_j^{(1)} \left(a S_+^{jj+1} - a^{\dagger} S_-^{jj+1}\right) \tag{7.145}$$

where

$$\alpha_j^{(1)} = \frac{g_j}{\Delta_{j+1} - \Delta_j} \tag{7.146}$$

is supposed to be a small number, i.e. $\alpha_j \ll 1$, which means that the atomic transitions are far from the one-photon resonance ($\Delta_{j+1} - \Delta_j = E_{j+1} - E_j - \omega_f \gg g_i$). Since $[T, h_0] = -V$, all one-photon transitions are eliminated by the transformation (7.145) and the transformed Hamiltonian takes the form

$$\begin{aligned} H_{\text{eff}}^{(1)} = U_1 H_{\text{int}} U_1^{\dagger} H_{\text{eff}}^{1} = h_0 + \sum_{p=1}^{N-2} \frac{p}{(p+1)!} \sum_{j=1}^{N-p-1} \psi_j^{(p+1)} \\ \times \left[a^{p+1} S_+^{jj+p+1} + a^{\dagger p+1} S_-^{jj+p+1}\right] + h_d + h_{nd} \end{aligned} \tag{7.147}$$

where the effective interaction constants ψ_j^p can be obtained from the recurrence relation

$$\psi_j^{(p+1)} = \alpha_{j+p}^{(1)} \psi_j^{(p)} - \alpha_j^{(1)} \psi_{j+1}^{(p)}, \quad \psi_j^1 = g_j \tag{7.148}$$

It is easy to see that $\psi_j^{(p)} \sim \alpha_j^{(p-1)}$ and thus $\psi_j^{(p+1)} \ll \psi_j^{(p)}$. The term h_d contains only diagonal terms on the atomic operators and depends on the integral of motion (or, equivalently, depends only on the photon number operator $a^{\dagger} a$). The operator h_d is naturally represented in a form of expansion on the small parameter $\alpha_j^{(1)}$ and the first term of this expansion has the form

$$h_d = \sum_{j=1}^{N-1} g_j \alpha_j^{(1)} \left[a^{\dagger} a \left(S^{j+1j+1} - S^{jj}\right) + (S^{jj} + 1) S^{j+1j+1}\right] + O(\alpha^3)$$

The operator h_{nd} contains only nondiagonal terms, which can be eliminated (if some specific resonant conditions, different from k-photon ones, are not fulfilled) by correspondent rotations of the type (7.145) in contrast to the diagonal operator h_d, which cannot be removed from the effective Hamiltonian (7.147). We note that the price paid for eliminating one-photon transitions is the generation of all possible (in this atomic configuration) k-photon transitions ($k = 2, \ldots, N-1$). The possibility of removing some of the terms in the sum appeared in Equation 7.147 strongly depends on the imposed resonant conditions.

As a nontrivial example, let us analyze the case of the three-photon resonance, and consider the interaction of a collection of A identical four-level atoms, $N = 4$, (in a cascade configuration) with a single-mode quantum field of frequency ω_f and suppose that there are no transitions in one- and two-photon resonances with the field.

After eliminating one-photon transitions, the transformed Hamiltonian (7.147) (written up to the corresponding order) has the form

$$\begin{aligned} H_{\text{eff}}^{(1)} &= h_0 + \frac{1}{2}\sum_{j=1}^{2} \psi_j^{(2)} \left[a^2 S_+^{jj+2} + a^{\dagger 2} S_-^{jj+2} \right] + \frac{1}{3}\psi_1^{(3)} \\ &\quad \times \left[a^3 S_+^{14} + a^{\dagger 3} S_-^{14} \right] + h_d + h_{nd} \end{aligned} \tag{7.149}$$

where

$$h_d = \sum_{j=1}^{3} g_j \alpha_j^{(1)} \left[a^\dagger a \left(S^{j+1j+1} - S^{jj} \right) + (S^{jj} + 1) S^{j+1j+1} \right] + O(\alpha^3)$$

and the essential part of the nondiagonal operator h_{nd} is

$$h_{nd} = \frac{1}{2}\sum_{i,j=1}^{3} \alpha_i^{(1)} g_j \left(S_+^{ii+1} S_-^{jj+1} + S_+^{jj+1} S_-^{ii+1} \right) + O(\alpha^3), \quad i \neq j \tag{7.150}$$

The interaction constants are defined according to Equations 7.146 and 7.148:

$$\begin{aligned} &\alpha_1^{(1)} = \frac{g_1}{\Delta_2}, \quad \alpha_2^{(1)} = \frac{g_2}{\Delta_3 - \Delta_2}, \quad \alpha_3^{(1)} = -\frac{g_3}{\Delta_3} \\ &\psi_1^{(2)} = g_1 g_2 \frac{2\Delta_2 - \Delta_3}{\Delta_2 (\Delta_3 - \Delta_2)}, \quad \psi_2^{(2)} = g_2 g_3 \frac{2\Delta_3 - \Delta_2}{\Delta_3 (\Delta_2 - \Delta_3)} \\ &\psi_1^{(3)} = \frac{3 g_1 g_2 g_3}{\Delta_3 \Delta_2}, \quad \Delta_2 = E_2 - E_1 - \omega_f, \quad \Delta_3 = E_3 - E_1 - 2\omega_f \end{aligned}$$

where the resonant condition (7.144) is $\Delta_4 = 0$ (meaning the three-photon resonance $E_4 - E_1 = 3\omega_f$) has been imposed. Note, that the term describing three-photon transitions is of order α^2.

According to the general scheme, the second term (representing two-photon transitions) in Equation 7.149 can be removed using a transformation analogous to the transformation (7.145):

$$U_2^{(1)} = \exp\left[T_2^{(1)} \right], \quad T_2^{(1)} = \frac{1}{2}\sum_{j=1}^{2} \alpha_j^{(2)} (a^2 S_+^{jj+2} - a^{\dagger 2} S_-^{jj+2}) \tag{7.151}$$

where

$$\alpha_j^{(2)} = \frac{\psi_j^2}{\Delta_{j+2} - \Delta_j}$$

is supposed to be a small parameter, which means that there are no resonant two-photon transitions ($\Delta_{j+2} - \Delta_j = E_{j+2} - E_j - 2\omega_f \gg \psi_j^{(2)}$) and thus $\alpha_j^{(2)} \ll \alpha_j^{(1)}$.

It is worth noting that the transformation (7.151) does not introduce new terms of order α^2 to the effective Hamiltonian.

The nondiagonal term h_{nd} (7.150), whose explicit form is

$$\begin{aligned} h_{nd} &= \frac{1}{2}\left(\alpha_1^{(1)} g_3 + \alpha_3^{(1)} g_1\right)\left(S_+^{12} S_-^{34} + S_+^{34} S_-^{12}\right) \\ &+ \frac{1}{2}\left(\alpha_1^{(1)} g_2 + \alpha_2^{(1)} g_1\right)\left(S_+^{12} S_-^{23} + S_+^{23} S_-^{12}\right) \end{aligned}$$

deserves more attention. It is clear that the first term in the above expression, under the condition

$$\Delta_2 = -\Delta_3 \tag{7.152}$$

describes a resonant dipole – dipole interaction. The second term describes a resonant interaction if

$$2\Delta_2 = \Delta_3 \tag{7.153}$$

which is incompatible with Equation 7.152 and conditions of absence of one- and two-photon resonances. Obviously, resonant interactions cannot be removed from the Hamiltonian. Nevertheless, if neither of the conditions given by Equations 7.152 and 7.153 is fulfilled, the term h_{nd} can be eliminated (in the given approximation) with the following transformation:

$$U_2^{(2)} = \exp\left[T_2^{(2)}\right], \quad T_2^{(2)} = \frac{1}{2}\sum_{i,j=1}^{3} \beta_{ij}\left(S_+^{ii+1} S_-^{jj+1} - S_+^{jj+1} S_-^{ii+1}\right), \quad i \neq j$$

where

$$\beta_{ij} = \frac{\alpha_i^{(1)} g_j}{\Delta_{i+1} - \Delta_i + \Delta_j - \Delta_{j+1}}$$

Then, taking into account (Equation 7.142) that $S^{11} + S^{22} + S^{33} + S^{44} = A$ and imposing the condition of the absence of initial population in levels 2 and 3, we obtain the effective Hamiltonian describing three-photon resonant transitions

$$\begin{aligned} H_{\text{eff}} &= \frac{g_1 g_2 g_3}{\Delta_2 \Delta_3}\left[a^3 S_+^{14} + a^{\dagger 3} S_-^{14}\right] \\ &- \left(S_3^{14} + \frac{A}{2}\right)\left[a^\dagger a\left(\frac{g_1^2}{\Delta_2} - \frac{g_3^2}{\Delta_3}\right) + \frac{g_3^2}{\Delta_3}\right] + A\frac{g_1^2}{\Delta_2} a^\dagger a \end{aligned} \tag{7.154}$$

where $S_z^{14} = (S^{44} - S^{11})/2$ and terms of order $1/\Delta^3$ have been omitted.

7.13
Problems

7.1 Find the exact evolution operator for the Dicke model in the case of two atoms and compare it with the approximate evolution operator in the case of a strong field.

7.2 Show that the emission from the following Dicke states is suppressed in the strong field case, $N \gg A$:

(a) $|\mathrm{in}\rangle = (|A,0\rangle + |A,A\rangle)/\sqrt{2}$;

(b) $|\mathrm{in}\rangle = |A, \frac{A}{2}\rangle$ (for a pair A).

Hint: Consider the following operator in the subspace with the number of excitations N:

$$\hat{\sigma} = \left| \begin{array}{cccc} 0 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{array} \right|, \qquad \sigma S_z \sigma = -S_z$$

Prove, that $\hat{\sigma}$ commutes approximately with the resonant Dicke Hamiltonian (6.13): $[\hat{\sigma}, V] \sim A/\sqrt{N}$.

7.3 Find the resonant fluorescence spectrum for the field in an initial number state with N photons and the atoms in an initial Dicke state with half of the atoms excited, $|A, A/2\rangle$. Show that the relative intensity of the central and side bands in this case is 4 : 1.

7.4 Find the effective Hamiltonian that describes the absorption–emission process for two different photons in a set of A three-level atoms in the cascade configuration. The interaction Hamiltonian is of the form

$$\begin{aligned} H = {} & \omega_a a^\dagger a + \omega_b b^\dagger b + E_1 S^{11} + E_2 S^{22} + E_3 S^{33} \\ & + g_1 \left(a S_+^{12} + a^\dagger S_-^{12} \right) + g_2 \left(b S_+^{23} + b^\dagger S_-^{23} \right) \end{aligned}$$

and the resonance conditions are $E_3 - E_1 = \omega_a + \omega_b$ and $E_3 - 2E_2 + E_1 \gg |\omega_a - \omega_b|$.

8

Quantum Systems Beyond the Rotating Wave Approximation

8.1

Kinematic and Dynamic Resonances in Quantum Systems

In a direct analogy with classical mechanics, composed systems in quantum optics (which describe interaction between several subsystems) can be divided into two classes: (i) systems that possess the necessary number of integrals of motion so that the classical counterpart is an integrable system; (ii) systems that do not admit a sufficient number of integrals of motions so that the classical counterpart is a nonintegrable system [91]. A quantum system may have basically two types of integrals of motion: (i) *kinematic* integrals, which do not depend on the kind of interaction between subsystems, as for instance, the total number of atoms; (ii) *dynamic* integrals, which are related to the particular form of interaction between subsystems, as, for instance, the number of excitations preserved (usually approximately) in some transitions between the energy levels of the subsystems.

Typical for quantum optical systems, dipole-like interactions between two subsystems (X and Y) can be described with a generic multichannel Hamiltonian (with time-independent couplings g_{jk}) of the following form:

$$H = \sum_j \omega_j X_0^{(j)} + \sum_k \Omega_k Y_0^{(k)} + \sum_{j,k} g_{jk} \left(X_-^{(j)} + X_+^{(j)} \right) \left(Y_-^{(k)} + Y_+^{(k)} \right) \tag{8.1}$$

where the two first terms represent the free Hamiltonians of the subsystems, so that the frequencies $\omega_j, \Omega_k \geq 0$, and the last term describes the interaction between them. The operators $X_{0,\pm}^{(j)}$, $Y_{0,\pm}^{(k)}$, $[X_{0,\pm}^{(j)}, Y_{0,\pm}^{(k)}] = 0$, satisfy the ladder commutation relations

$$\left[X_0^{(j)}, X_\pm^{(j)} \right] = \pm X_\pm^{(j)}, \quad \left[Y_0^{(j)}, Y_\pm^{(j)} \right] = \pm Y_\pm^{(j)} \tag{8.2}$$

but do not necessarily close a finite-dimensional Lie algebra and could be elements of some deformed algebra [81, 92].

In the interaction Hamiltonian, there are two kinds of terms: of the forms $X_-^{(j)} Y_+^{(k)}$ and $X_-^{(j)} Y_-^{(k)}$. It is easy to observe that in the rotating frame, that is, applying the

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unitary transformation

$$U = \exp\left(it\left[\sum_j \omega_j X_0^{(j)} + \sum_k \Omega_k Y_0^{(k)}\right]\right)$$

to the Hamiltonian (8.1), the *counter-rotating* terms $\sim X_-^{(j)} Y_-^{(k)}$ oscillate in time with a frequency $\omega_j + \Omega_k$ and the *rotating* terms $\sim X_-^{(j)} Y_+^{(k)}$ oscillate with a frequency $\omega_j - \Omega_k$. It is clear that under the condition $\omega_j \approx \Omega_k$, the rotating term $X_-^{(j)} Y_+^{(k)}$ in Equation 8.1 is approximately time independent (and thus, can generate transitions with a probability of one between the energy levels of the system); meanwhile, the counter-rotating term, $X_-^j Y_-^k$ always oscillates rapidly in the rotating frame, and its temporal average is zero.

By neglecting the counter-rotating terms in the Hamiltonian (8.1), which is commonly called the *rotating wave approximation* (*RWA*), we arrive at the Hamiltonian

$$H = \sum_j \omega_j X_0^{(j)} + \sum_k \Omega_k Y_0^{(k)} + \sum_{j,k} g_{jk}\left(X_-^{(j)} Y_+^{(k)} + X_+^{(j)} Y_-^{(k)}\right) \tag{8.3}$$

which admits several dynamic integrals of motion N_p, $p = 1, 2, \ldots n$, generally not allowed in Equation 8.1. This implies that the whole representation space of the system is divided into finite-dimensional invariant subspaces, and the mathematical treatment is essentially simplified.

It is worth noting, that the semiclassical models, where some of the subsystems are described by *c*-numbers instead of operators, are treated using essentially the same type of Hamiltonians as in Equation 8.1. The semiclassical transition in Equation 8.1 can be done by going to the rotating frame of the semiclassical system and then, just substituting the transition operators by some complex numbers. For instance, in the case of single channel interaction, $j = 1$, $k = 1$, when the system Y acquires classical features, the Hamiltonian (8.1) in the rotating frame corresponding to the system Y takes the form

$$H = \omega X_0 + g\,(X_- + X_+)\left(Y_- e^{i\Omega t} + Y_+ e^{-i\Omega t}\right)$$

so that the corresponding semiclassical Hamiltonian is obtained by substituting $Y_+ \to g$, $Y_- \to g^*$, giving

$$H_{\mathrm{sc}} = \omega X_0 + g\,(X_- + X_+)\left(g^* e^{i\Omega t} + g e^{-i\Omega t}\right) \tag{8.4}$$

Such Hamiltonians usually appear when a quantum oscillator and/or a collection of atoms is pumped by an external force [22, 61].

In the RWA-like systems, described by Hamiltonians of the form (8.3), the resonance conditions

$$\omega_j \approx \Omega_k \tag{8.5}$$

mean that the term $X_-^{(j)} Y_+^{(k)}$, which explicitly appears in the Hamiltonian, does not depend on time in an appropriate rotating frame. Nevertheless, such *explicit* *resonances* are not the only kind of resonances that can be found in the Hamiltonian (8.3). Usually, the composed systems admit several types of *implicit resonances* related to effective transitions between their energy levels, which do not appear in the original Hamiltonian. Such effective interactions play important roles in many physical applications and can be revealed by adiabatic elimination of slow transitions [93]. The implicit (effective) resonances are characterized by their position, $\sum_j m_j\omega_j \approx \sum_k n_k\Omega_k$, and strength, i.e. in what order of perturbation expansion they appear for the first time. Although, a generic system can possess a large number of different types of effective transitions, all the possible resonance conditions can be classified only by analyzing the free Hamiltonian and the integrals of motion (explicit form of which depends on the symmetry of the interacting systems). Because all the invariant subspaces are finite dimensional, there are always a finite number of different resonances independent of the nature of interacting systems, even when some of the interacting systems are infinite dimensional (quantized fields). We refer to these kinds of resonances as *kinematic resonances*; they include both explicit and implicit resonances.

In Section 8.2, we show with the example of atom–field interactions, that it is possible to classify all the kinematic resonances in a straightforward way. It is worth nothing that for not very strong interactions, each resonance can be isolated in the interaction Hamiltonian in such a way that a specific term would describe a given resonance transition.

The situation is quite different in quantum systems with a lack of integrals of motion, corresponding to classically nonintegrable dynamic systems. It is worth noting that the Hamiltonian (8.1) still describes an integrable system when both X and Y systems are harmonic oscillators. Such systems are frequently called *linear* (see Appendix 11.3) because the Hamiltonian governing their evolution can be represented as a linear form on the generators of some finite-dimensional Lie algebra. In such cases, the Heisenberg equations of motion are linear, so that the average values evolve according to the classical Hamiltonian equations (or in other words, exactly follow the classical trajectories). In what follows, we focus on nonlinear systems, especially atom–field interactions.

From the general point of view, we can discriminate at least three interesting limits: (i) when the interaction constants between the subsystems g_{jk} is much higher than the characteristic frequencies of the interacting systems; (ii) when g_{jk} is much smaller than the frequencies of the systems; and (iii) when g_{jk} is larger than or of the order of the frequency of some of the interacting systems, but much smaller than the frequency of the others, so that we can divide the whole system into slow and fast (whose transition frequency is much larger than all the other characteristic frequencies of the system) interacting subsystems.

Case (i), which corresponds to very strong coupling should be studied carefully because using the expansion parameter like an interaction constant over a characteristic frequency could be quite tricky. For instance, the type of the spectrum corresponding to the nonperturbed and to the perturbed systems can be different: either continuous or discrete.

In case (ii), an infinite number of different resonances arise and a priori it is impossible to determine their position and strength, which essentially depend not only on the type of interaction but also on the algebraic properties of each interacting subsystem [25,94,95]. In Section 8.3, we discuss such *dynamic resonances*, analyzing different models of interaction of quantum and classical fields with atomic systems. We show that, in this case, the evolution is governed by an effective Hamiltonian describing a certain resonant interaction and the representation space of the total system can be always divided into (almost) invariant subspaces.

The last case, case (iii) possesses a peculiar property: besides finding a corresponding effective Hamiltonian for the slow subsystem, we can also project it out into the lower energy state of the fast system, which would never get excited under given relations between the system's parameters, and thus, describe an effective dynamics of the slow system in the limit of strong interaction. Several interesting phenomena, in particular, the possibility of quantum phase transition (QPT) [125] can be observed in this region of the system's parameters.

In the limit of weak $g_{jk} \ll \omega_n, \Omega_m$ and intermediate $\omega_n \lesssim g_{jk} \ll \Omega_m$ (or $\Omega_m \lesssim g_{jk} \ll \omega_n$) interactions, it is convenient to analyze the system's behavior using the effective Hamiltonian approach, which can be obtained by applying, for instance, the Lie transformation method outlined in the Appendix 11.4. The main idea consists in removing the counter-rotating terms from the Hamiltonian (8.1) using the method of adiabatic elimination. Although, such analysis can be performed for a general system, we focus on the simplest case of a one channel Hamiltonian and show that, even in this case, the absence of the integral of motion corresponding to the total number of excitations leads to the appearance of a series of (dynamic) resonances in the weak interaction limit (which can be classified according the interaction type) and QPTs, in the intermediate limit.

8.2
Kinematic Resonances: Generic Atom–Field Interactions

Since the principal interest represents the multichannel processes and the analysis of generic systems becomes very cumbersome, we will focus exclusively on the kinematic resonances in the case of atom–field interactions.

Let us consider the interaction of a collection of A identical N-level atoms of an arbitrary configuration with a single mode of a quantized field of frequency ω. The Hamiltonian describing this system has the form

$$H = H_d + H_{\text{int}} \tag{8.6}$$

where

$$H_d = \omega a^\dagger a + \sum_{j=1}^{N} E_j S^{jj}, \quad E_j < E_{j+1} \tag{8.7}$$

is the free Hamiltonian, and S^{jj} are the collective atomic population operators corresponding to the j-th atomic level of energy E_j, and H_{int} is the interaction atom–field Hamiltonian, whose explicit form depends on the atomic configuration in the dipole approximation, i.e. only one-photon transitions are allowed.

In this section, we suppose that the RWA is imposed, so that the total number of excitations in the atom–field system is preserved, and thus, the whole representation space of this quantum system is divided into finite-dimensional invariant subspaces.

The Hamiltonian (8.6) admits two integrals of motion: a kinematic integral, given by the total number of atoms

$$A = \sum_{i=1}^{N} S^{ii} \tag{8.8}$$

and a dynamic integral, corresponding to the total number of excitations in the system

$$N = a^{\dagger}a + \sum_{i=1}^{N} \mu_j S^{ii} \tag{8.9}$$

where the parameters μ_j depend on the atomic configuration.

We note that a *generic* interaction term can be written as follows:

$$f(S^{ii}, a^{\dagger}a) \prod_{j=1}^{N-1} \left(S_{+}^{jN}\right)^{k_j} a^{k_N}, \quad k_j \in \mathcal{Z},\ j = 1, \dots, N \tag{8.10}$$

where $S_{\pm}^{ij}$ ($S_{+}^{ij} = S^{ij}, S_{-}^{ij} = S^{ji}, j > i$) are the atomic transition operators satisfying the $u(N)$ commutation relations, $\left[S^{ij}, S^{km}\right] = \delta_{im}S^{kj} - \delta_{kj}S^{im}$, and negative exponents correspond to the Hermitian conjugated operators. The operational coefficients $f(S^{ii}, a^{\dagger}a)$ depend on diagonal atomic operators and the photon number operator. In what follows, we omit the coefficient $f(S^{ii}, a^{\dagger}a)$, since it leads only to some phase shifts and does not change the distribution of excitations in the system's energy levels. It is worth noting that although Equation 8.10 is not a unique way to represent a generic interaction term, any other interactions preserving Equations (8.8) and (8.9) can be written down as a product as in Equation 8.10.

Since the RWA is imposed, every interaction term should preserve the total number of excitations, so the condition

$$\left[N, \prod_{j=1}^{N-1} \left(S_{+}^{jN}\right)^{k_j} a^{k_N}\right] = 0$$

is held, and thus, the numbers k_j satisfy the following restriction:

$$k_N = \sum_{j=1}^{N-1} k_j \left(\mu_N - \mu_j\right) \tag{8.11}$$

Thus, any admissible interaction can be described as

$$\prod_{j=1}^{N-1} \left(S_{+}^{jN}\right)^{k_j} a^{\sum_{j=1}^{N-1} k_j\left(\mu_N - \mu_j\right)} \tag{8.12}$$

The interaction given by Equation 8.12 becomes resonant when the atomic transition energies and the field frequency satisfy the following condition

$$\sum_{j=1}^{N-1} k_j \left(E_N - E_j - \omega\left(\mu_N - \mu_j\right)\right) = 0 \tag{8.13}$$

The important point here is that interaction terms in Equation 8.12 can be explicitly presented as in Equation 8.6, or can describe effective interactions, and thus should be obtained from the original Hamiltonian by adiabatic elimination of some far-off resonant transitions. Thus, the condition (8.13) describes *explicit* resonances if the corresponding interaction term is present in the original Hamiltonian or *implicit* resonances, if such interaction is effective. It is clear that the total number of both explicit and implicit resonances is finite, which is a consequence of the restriction (Equation 8.11) imposed by the RWA.

The number of possible resonances depends on the number of atomic levels, N, and the total number of atoms, A. Note that the resonance condition (8.13) is associated with a vector $\mathbf{k} = \left(k_1, \ldots k_{N-1}\right)$; to avoid repetition of the resonances, we consider only vectors $\mathbf{k}$ with coprime components (that is, $k_1, \ldots k_{N-1}$ do not have a common factor). Then, different vectors $\mathbf{k}$ satisfying the condition

$$\max\left\{\left|\sum_{i=1}^{N-1} k_i\right|, \left|\sum_{i=1, j\neq i}^{N-1} k_i\right|_{j=1,\ldots,N-1}, \ldots, \left|k_i\right|_{i=1,\ldots,N-1}\right\} \leq A \tag{8.14}$$

define different resonances.

On the other hand, taking into account the dynamic integral of motion (8.9), one can rewrite the free Hamiltonian (8.7) as follows:

$$H_d = \omega N + \sum_{j=1}^{N} \left(E_j - \mu_j \omega\right) S^{jj} \tag{8.15}$$

Summing the kinematic integral $-\beta \sum_{i=1}^{N} S^{ii}$ (Equation 8.8) to the above Hamiltonian, where the constants $\beta = \beta(\mathbf{n}), \mathbf{n} = (n_1, \ldots, n_N)$, are chosen such that

$$\beta(\mathbf{n}) = \sum_{j=1}^{N} n_j \left(E_j - \mu_j \omega\right), \quad \sum_{j=1}^{N} n_j = 1, \quad n_j \in \mathcal{Z}$$

we obtain

$$H_d = \omega N + \sum_{j=1}^{N} \left(E_j - \mu_j \omega - \beta(\mathbf{n})\right) S^{jj} \tag{8.16}$$

It is easy to see that the condition $E_j - \mu_j \omega - \beta(\mathbf{n}) = 0$, for any fixed j and some values of $\beta(\mathbf{n})$, enumerates all the possible resonances (8.13). Let us take, for instance, $j = 1$, if $E_1 - \mu_1 \omega - \beta(\mathbf{n}) = 0$, then

$$\sum_{j=2}^{N-1} n_j \left(E_N - E_j - \omega\left(\mu_N - \mu_j\right)\right) + (n_1 - 1)\left(E_N - E_1 - \omega\left(\mu_N - \mu_1\right)\right) = 0$$

which coincides with Equation 8.13, when $k_j = n_j$ for $j = 2, \ldots N - 1$ and $k_1 = (n_1 - 1)$.

This means that we can always represent the free Hamiltonian in a way that all the possible resonance conditions, corresponding to both explicit and implicit resonances, appear as zeros of coefficients of the atomic population operators S^{jj} in the free Hamiltonian, after taking out the integral of motion (8.9) corresponding to the total number of excitations.

The above allows us to classify all the possible kinematic resonances:

1. *Multiphoton resonances:* transitions that involve absorption and emission of photons: (i) simple n-photon transitions, described by the terms $\sim a^n S_+^{kj}$, with the resonance condition $E_j - E_k \approx n\omega$. The terms with $n = 1$ are usually present in the original Hamiltonian and, in such cases, represent explicit resonances. Simple n-photon transitions appear even in a single atom case; (ii) collective atomic transitions, described by the terms $\sim a^n S_+^{kl} S_-^{ji}$, (in general, a product of several – up to A – atomic transition operators can appear), with the corresponding resonance condition $(E_l - E_k) - \left(E_i - E_j\right) \approx n\omega$. It is clear that such resonances can appear only in multiatom systems. Note, that if $l = i$ (or $k = j$), then we obtain the same resonance condition as in (i), describing an effective process of absorption of n photons with atomic transition from kth to jth energy levels. Nevertheless, the corresponding term in the effective Hamiltonian would be multiplied by the atomic population operator S^{ll}, which means that more than one atom is needed to realize such a process.
2. *Virtual photon resonances*: atomic transitions between independent channels caused by the quantum field fluctuations, and thus existing even when the field is in the vacuum state. Such resonances appear only when the system has more than one atom and are described by terms in the form of products of atomic transitions operators. For instance, the simplest term of this kind (typically appearing in the first-order perturbation expansion) is $S_+^{kj} S_-^{il}$, $k \neq i$, $j \neq l$, describes atomic transitions $k \to j$, $l \to i$, and the corresponding resonance condition is

   $$\left(E_j - E_k\right) - (E_l - E_i) \approx 0 \qquad (8.17)$$

   Obviously, such transitions can be realized in a system that consists of at least two atoms. Note that the term $\sim S_+^{lj} S_-^{ik}$, which represents the atomic transition $k \to i$, $l \to j$, satisfies the same resonant condition. More involved interactions, like $\sim S_+^{ij} S_-^{kl} S_-^{mn}$, can appear in the highest orders of the perturbation theory. It is worth noting that the strength of virtual photon transitions does not depend on the field intensity in the leading order of the perturbation expansion.
3. *Photon-assisted transitions*: atomic transitions when every photon emission is accompanied by a simultaneous photon absorption. These transitions appear only when the atomic configuration contains coherent channels, similar to lambda-like configurations. The strength of such interactions depends on the

number of photons in the field and the populations of some atomic energy levels. The simplest terms describing the photon-assisted transition (which appear in the lowest order of the perturbation theory) have the form: $f(a^\dagger a, S^{jj})S_+^{kl}$, corresponding to the resonance condition $E_l \approx E_k$, where the transition $k \leftrightarrow l$ is not present in the original Hamiltonian and, in the leading order, $f(a^\dagger a, S^{jj})$ is a linear polynomial of the photon number operator and the atomic population operators.

In the above classification, we do not consider interactions corresponding to powers of terms as they appear in perturbation expansion. For instance, the transition corresponding to the term $(aS_+^{jk})^n$, that describes an absorption of n photons by n atoms in the transition $j \rightarrow k$ belongs to "one-photon transitions".

In the case of interaction of an atomic system with a classical field, the RWA implies that the interaction Hamiltonian can always be reduced to a time-independent form. In this case, apart from explicit resonances, several types of implicit (effective) resonant transitions can take place. It is clear that no interactions similar to virtual photon resonances can arise. Nevertheless, transitions similar to multiphoton resonances $\sim S_+^{jk}, k-j \geq 2$ (and photon-assisted resonances), where the transition $k \leftrightarrow j$ does not exist in the initial Hamiltonian, actually appear.

As a simple, but representative example of the kinematic resonances, we analyze the interaction of A four-level diamond configuration atoms with a single mode of a quantized field under RWA (Figure 8.1). The Hamiltonian governing the evolution of this system has the form

$$H = \omega a^\dagger a + \sum_{j=1}^{4} E_j S^{jj} + g_1\left(aS_+^{12} + a^\dagger S_-^{12}\right) + g_2\left(aS_+^{13} + a^\dagger S_-^{13}\right) \\ + g_3\left(aS_+^{24} + a^\dagger S_-^{24}\right) + g_4\left(aS_+^{34} + a^\dagger S_-^{34}\right) \tag{8.18}$$

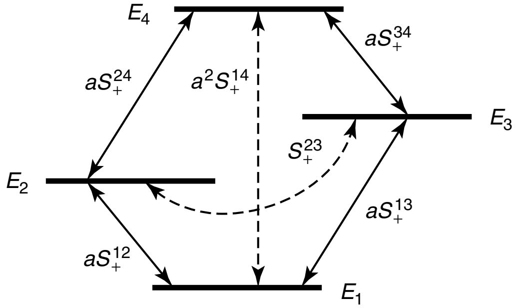

Figure 8.1 Diagram of interactions for a single four-level atom of diamond configuration. Solid line: explicit resonances; dashed line: implicit resonances; two-photon transition, photon-assisted transition.

This Hamiltonian describes four one-photon atomic transitions, ($1 \leftrightarrow 2$, $1 \leftrightarrow 3$), ($2 \leftrightarrow 4$, $3 \leftrightarrow 4$), gathered in two pairs of coherent quantum channels. The corresponding resonance conditions (explicit resonances) are $E_2 - E_1 \approx \omega$, $E_3 - E_1 \approx \omega$, $E_4 - E_2 \approx \omega$, and $E_4 - E_3 \approx \omega$. To find all the other (implicit) resonances, we follow the outlined method above. The dynamic integral of motion for this system is

$$N = a^\dagger a + S^{44} - S^{11}$$

so that the coefficients μ_j in Equation 8.9, are $\mu_1 = -1$, $\mu_2 = \mu_3 = 0$, $\mu_4 = 1$. Substituting these μ_j into Equation 8.13, we obtain all the possible resonance conditions:

$$k_1 \left(E_4 - E_1 - 2\omega\right) + k_2 \left(E_4 - E_2 - \omega\right) + k_3 \left(E_4 - E_3 - \omega\right) \approx 0 \tag{8.19}$$

$k_1, k_2, k_3 \in \mathcal{Z}$, and the corresponding (effective) interactions are

$$\left(S_+^{14}\right)^{k_1} \left(S_+^{24}\right)^{k_2} \left(S_+^{34}\right)^{k_3} a^{k_3+k_2+2k_1}$$

The vector $\mathbf{k} = \left(k_1, k_2, k_3\right)$ associated with possible resonances should satisfy the condition (8.14)

$$\max\left\{\left|\sum_{i=1}^{3} k_i\right|, \left|k_i + k_j\right|_{i \neq j}, \left|k_i\right|\right\} \leq A, \quad i, j = 1, 2, 3$$

For a single atom, we have the vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, -1, 0)$, $(1, 0, -1)$, and $(0, 1, -1)$, which correspond to the following resonance conditions (interactions): $E_4 - E_1 \approx 2\omega \left(a^2 S_+^{14}\right)$, $E_4 - E_2 \approx \omega \left(a S_+^{24}\right)$, $E_4 - E_3 \approx \omega \left(a S_+^{34}\right)$, $E_2 - E_1 \approx \omega \left(a S_+^{12}\right)$, $E_3 - E_1 \approx \omega \left(a S_+^{13}\right)$, and $E_3 \approx E_2 \left(S_+^{23}\right)$, respectively. For two atoms, there are 15 other vectors apart from the 6 shown above.

For a better understanding of the nature of implicit resonances, we find the first-order effective Hamiltonian considering that all the interactions appearing in the initial Hamiltonian (8.18) are far from resonance. Following the method outlined in the Appendix 11.4. and taking into account that the explicit transitions are far from the resonance, we apply to the Hamiltonian (8.18) the following sequence of unitary transformations:

$$H_{\text{eff}} = U_4 U_3 U_2 U_1 H U_1^\dagger U_2^\dagger U_3^\dagger U_4^\dagger$$

where $U_i = \exp\left(\varepsilon_i A_i\right)$,

$$A_1 = a S_+^{12} - a^\dagger S_-^{12}, \quad A_2 = a S_+^{13} - a^\dagger S_-^{13}$$

$$A_3 = a S_+^{24} - a^\dagger S_-^{24}, \quad A_4 = a S_+^{34} - a^\dagger S_-^{34}$$

and the small parameters $\varepsilon_i \ll 1$ are given by

$$\varepsilon_1 = \frac{g_1}{E_2 - E_1 - \omega}, \quad \varepsilon_2 = \frac{g_2}{E_3 - E_1 - \omega}$$

$$\varepsilon_3 = \frac{g_3}{E_4 - E_2 - \omega}, \quad \varepsilon_4 = \frac{g_4}{E_4 - E_3 - \omega}$$

We obtain, in the first order in small parameters, the following effective Hamiltonian

$$\begin{aligned} H = \omega a^\dagger a + \sum_{j=1}^{4} E_j S^{jj} + \Phi(a^\dagger a, S^{jj}) + g_2\varepsilon_1 \left[S_+^{23} \left(S^{11} + a^\dagger a + 1 \right) + h.c. \right] \\ + g_4\varepsilon_3 \left[S_+^{23} \left(S^{44} - a^\dagger a \right) + h.c. \right] - \left(g_3\varepsilon_1 + g_4\varepsilon_2 \right) \left(a^2 S_+^{14} + h.c. \right) \\ + g_3\varepsilon_1 \left(S_+^{24} S_-^{12} + h.c. \right) + g_3\varepsilon_2 \left(S_+^{24} S_-^{13} + h.c. \right) \\ + g_4\varepsilon_1 \left(S_+^{12} S_-^{34} + h.c. \right) + g_4\varepsilon_2 \left(S_+^{13} S_-^{34} + h.c. \right) \end{aligned} \tag{8.20}$$

where

$$\begin{aligned} \Phi(a^\dagger a, S^{jj}) = g_1\varepsilon_1 \left(S^{22} \left(S^{11} + 1 \right) + a^\dagger a \left(S^{22} - S^{11} \right) \right) \\ + g_2\varepsilon_2 \left(S^{33} \left(S^{11} + 1 \right) + a^\dagger a \left(S^{33} - S^{11} \right) \right) \\ + g_3\varepsilon_3 \left(S^{44} \left(S^{22} + 1 \right) + a^\dagger a \left(S^{44} - S^{22} \right) \right) \\ + g_4\varepsilon_4 \left(S^{44} \left(S^{33} + 1 \right) + a^\dagger a \left(S^{44} - S^{33} \right) \right) \end{aligned}$$

is the dynamic Stark shift [62].

Let us classify the effective resonances present in the above (first order) Hamiltonian:

1. *Multiphoton resonances*: the only two resonances of this type are one-photon and two-photon resonances. There are four one-photon transitions corresponding to explicit resonances, which do not appear in the effective Hamiltonian (8.20). The term $\sim a^2 S_+^{14}$ describes two-photon transitions, with corresponding resonance condition $E_4 - E_1 \approx 2\omega$ and $\mathbf{k} = (1, 0, 0)$.
2. *Virtual photon resonances:* these correspond to the last four terms in Equation 8.20, $\sim S_+^{24} S_-^{12},\ S_+^{24} S_-^{13},\ S_+^{12} S_-^{34},\ S_+^{13} S_-^{34}$, with the resonance conditions $E_4 - E_2 \approx E_2 - E_1,\ E_4 - E_2 \approx E_3 - E_1,\ E_2 - E_1 \approx E_4 - E_3,$ and $E_3 - E_1 \approx E_4 - E_3$ respectively.
3. *Photon-assisted resonances:* the only resonance of this type is of the form $\sim f\left(S^{44}, S^{11}, a^\dagger a\right) S_+^{23}$; it is produced in the first order of the perturbation theory generating transitions between the middle atomic levels, and the corresponding resonance condition is $E_3 \approx E_2$.

8.3
Dynamic Resonances

In this section, we study quantum systems corresponding to classically nonintegrable systems, mainly focusing on the models describing interaction of atoms with quantized and classical fields without applying the RWA [25, 94, 95] in the limit of weak interaction, $g_{jk} \ll \omega_n, \Omega_m$. The main interesting features of such systems can be studied in the particular case of single-channel Hamiltonians (8.1) of type,

$$H = \omega X_0 + \Omega Y_0 + g(X_+ Y_- + X_- Y_+) + g\,(X_+ Y_+ + X_- Y_-) \tag{8.21}$$

where X_0 (Y_0) is the free Hamiltonian of the subsystem X (Y) and X_+ (Y_+), X_- (Y_-) are the raising and lowering operators, respectively, which describe transitions between the energy levels in the subsystem X (Y), and satisfy the commutation relations (8.2). We do not impose any commutation relation between the transition operators, which are generally some functions of the diagonal operators and integrals of motion corresponding to the systems X and Y, $[N_1, X_{0,\pm}] = [N_2, Y_{0,\pm}] = 0$,

$$[X_+, X_-] = P_1\,(X_0) = \nabla_{X_0}\phi_1\,(X_0, N_1)$$
$$[Y_+, Y_-] = P_2\,(Y_0) = \nabla_{Y_0}\phi_2\,(Y_0, N_2)$$

where $\phi_1\,(X_0, N_1)$, $\phi_2\,(Y_0, N_2)$ are given by the structural functions

$$\phi(X_0) = X_+ X_- \tag{8.22}$$

and

$$\nabla_z \phi\,(z) = \phi\,(z) - \phi\,(z+1)$$

The consequences of the existence of the *counter-rotating* term $\sim X_+ Y_+$ in the Hamiltonian (8.21) are (i) the dimension of the whole representation space is the product of the corresponding dimensions of the subsystems: $\dim(X \otimes Y) = \dim X \cdot \dim Y$; (ii) there are some additional resonances apart from $\omega = \Omega$, that do not exist in the case of a single-channel Hamiltonian under RWA (Equation 8.3); (iii) the type of resonances depends on the structure of the algebras describing the X and Y systems.

The counter-rotating term rapidly oscillates (with frequency $\omega + \Omega$), and thus, in the weak interaction limit, $\omega, \Omega \gg g$, it can be eliminated by applying the Lie-type transformation

$$U = \exp\left[\varepsilon\,(X_+ Y_+ - X_- Y_-)\right], \quad \varepsilon = \frac{g}{(\omega + \Omega)} \ll 1 \tag{8.23}$$

to the Hamiltonian (8.21) and using the standard perturbative expansion (see Appendix 11.4). From now on, we suppose that $\omega \geq \Omega$.

The elimination of the term $X_+ Y_+ + X_- Y_-$ leads to the appearance of new elements in the transformed Hamiltonian, $H_1 = U_1 H U_1^\dagger$, which acquires the following form:

$$H_1 = \omega_1 X_0 + \omega_2 Y_0 + \varepsilon g \sum_{n=0}^{\infty}{}' \varepsilon^n \left[(X_+ Y_+)^n\, F_n(X_0, Y_0, \varepsilon) + h.c\right] + \tag{8.24}$$
$$g \sum_{n=0}^{\infty} \varepsilon^n \left[X_+ Y_- \left((X_+ Y_+)^n\, H_n(X_0, Y_0, \varepsilon) + (X_- Y_-)^n\, G_n(X_0, Y_0, \varepsilon)\right) + h.c.\right]$$

where $F_n(X_0, Y_0, \varepsilon)$, $H_n(X_0, Y_0, \varepsilon)$, $G_n(X_0, Y_0, \varepsilon)$ are diagonal operators that can be represented as series in a small parameter ε:

$$F_n(X_0, Y_0, \varepsilon) = \frac{2\,(n+1)}{(n+2)!} \nabla^n_{X_0,Y_0} \Phi\,(X_0, Y_0) + O(\varepsilon)$$

$$\Phi\,(X_0, Y_0) = \phi_1\,(X_0)\,\phi_2(Y_0) - \phi_2\,(Y_0+1)\,\phi_1(X_0+1) + O(\varepsilon)$$

$$H_n(X_0, Y_0, \varepsilon) = \frac{1}{n!} \frac{\nabla^n_{Y_0} \phi_2(Y_0)}{\phi_2(Y_0+n)} + O(\varepsilon)$$

$$G_n(X_0, Y_0, \varepsilon) = \frac{1}{n!} \frac{\nabla^n_{X_0} \phi_1(X_0-n+1)}{\phi_1(Y_0-n+1)} + O(\varepsilon)$$

and

$$\nabla_{X_0,Y_0} \Phi\,(X_0, Y_0) = \Phi\,(X_0, Y_0) - \Phi\,(X_0+1, Y_0+1)$$

the prime ($'$) in the first sum in Equation 8.24 means that the term with $n = 0$ is taken with the coefficient 1/2.

The Hamiltonian (8.24) still has quite complicated structure and can be simplified under certain resonance conditions. All the new counter-rotating terms, like $\sim X_+^n Y_+^m$, can be eliminated by applying the Lie-type transformation

$$U = \exp\left[\pi(\varepsilon_1 \cdots \varepsilon_k)\eta\left(X_+^n Y_+^m - X_-^n Y_-^m\right)\right] \tag{8.25}$$

where $\eta \approx g/\,(n\omega + m\Omega) \ll 1$, and $\pi(\varepsilon_1 \cdots \varepsilon_k)$ is the product of powers of some small parameters $\varepsilon_j \ll 1$, $j = 1, \ldots, k$. In fact, $\pi(\varepsilon_1 \cdots \varepsilon_k)$ is proportional to the coefficient of the term $\sim X_+^n Y_+^m + X_-^n Y_-^m$. In particular, it is easy to observe that all the terms of the form

$$a_n\left((X_+ Y_+)^n\, F_n(X_0, Y_0, \varepsilon) + h.c.\right) \tag{8.26}$$

can be eliminated (in a leading order on ε) by applying transformations

$$U_{11} = \exp\left[\frac{a_n}{n}\left((X_+ Y_+)^n\, F_n(X_0, Y_0, \varepsilon) - h.c.\right)\right], \quad n \geq 1 \tag{8.27}$$

where $a_n \sim \varepsilon^{n+1}$. Similarly, the terms

$$X_+ Y_-\left((X_+ Y_+)^n\, H_n(X_0, Y_0, \varepsilon) + (X_- Y_-)^n\, G_n(X_0, Y_0, \varepsilon)\right) + h.c.$$

can be removed from Equation 8.24. The elimination of the above terms leads to the appearance of new elements in the transformed Hamiltonian. The new terms can be divided into three groups: the first group contains terms that can always be eliminated under the condition $\omega, \Omega \gg g$ by applying some suitable transformations; the second group consists of terms that cannot be removed if certain relations between ω and Ω hold, since the rotation that eliminates a given term from the Hamiltonian becomes singular. This group contains certain nondiagonal terms of the form $X_+^n Y_-^m + h.c.$, which describe transitions between energy levels of the whole system. For example, the term $X_+Y_- + h.c.$ cannot be eliminated from the Hamiltonian if $\omega \approx \Omega$ because the corresponding transformation, having the form $\exp\left[g\left(X_+Y_- - h.c._-\right)/(\omega-\Omega)\right]$, becomes singular. The third group includes the diagonal terms (functions only of X_0, Y_0). In the Hamiltonian, we retain only the diagonal terms and those that cannot be eliminated by nonsingular transformations (resonant terms). Besides, we conserve only the leading order coefficients in these terms. The resonant terms appear as a consequence of transformation of the terms $X_+Y_-\left(X_+Y_+\right)^k H_k(X_0, Y_0, \varepsilon) + h.c.$ with

$$U_{2n} = \exp\left[\frac{g\varepsilon^n}{(n-1)\,\omega + (n+1)\,\Omega}\left(X_+Y_-\left(X_-Y_-\right)^n G_n(X_0, Y_0, \varepsilon) - h.c.\right)\right] \tag{8.28}$$

(which is applied to eliminate terms $X_+Y_-\left(X_-Y_-\right)^n G_n(X_0, Y_0, \varepsilon) + h.c.$). The leading order terms come from the transformations (8.28) with $n = 1$ and by taking only the leading terms in the expansions

$$U_{21} = \exp\left[\delta\epsilon\left(Y_-^2\nabla_{X_0}\phi_1(X_0) - h.c.\right)\right]$$

where $\delta = g/2\Omega$.

As a result, we obtain the following effective Hamiltonian:

$$H_{\text{eff}} \approx \omega X_0 + \Omega Y_0 + g\varepsilon\Phi\left(X_0, Y_0, \varepsilon\right) +$$
$$+g\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-\delta)^{l+k-1}\varepsilon^{l+2(k-1)}}{(k-1)!\left(l+k-1\right)!}\left[X_+^k Y_-^{2l+k}\theta_{kl}(X_0, Y_0, \varepsilon) + h.c.\right] \tag{8.29}$$

The term $\Phi\left(X_0, Y_0, \varepsilon\right)$ represents the dynamic Stark shift (or the Bloch–Siegert shift) [60] and can be expanded in powers of the small parameter ε as

$$\Phi\left(X_0, Y_0, \varepsilon\right) = \phi_1\left(X_0\right)\phi_2(Y_0) - \phi_1\left(X_0+1\right)\phi_2(Y_0+1) + O(\varepsilon^2)$$

The terms of the form $X_+^k Y_-^{2l+k} + h.c.$ describe all the admissible resonant interactions. In particular, the term with $k = 1$, $l = 0$ represents the principal resonance. The coupling constants $\theta_{kl}(X_0, Y_0, \varepsilon)$ depend on the algebraic structure of the operators describing both subsystems (more precisely, on the degree of the structural polynomials $\phi_1\left(X_0, N_1\right)$ and $\phi_2\left(Y_0, N_2\right)$), and in particular, they can become zero for some k, l. The coefficient in the principal term has the form

$$\theta_{10}(X_0, Y_0) = 1$$

and

$$\theta_{kl}(X_0, Y_0) = \sum_{j=0}^{l+[k/2]} C_{l+[k/2]}^{j}\left(P_1\left(X_0+k\right)\right)^{l+[k/2]-j}\left(-P_1\left(X_0\right)\right)^j R_k(X_0, Y_0-2j)$$

$$R_k(X_0, Y_0) = \sum_{j=0}^{[\frac{k-1}{2}]} C^j_{[\frac{k-1}{2}]} \left(P_1\left(X_0 + k\right)\right)^{[\frac{k-1}{2}]-j} \left(-P_1\left(X_0\right)\right)^j \nabla^{k-1}_{Y_0} \phi_2\left(Y_0 - 2j\right)$$
$$\times \prod_{i=1-2j}^{k-2(j+1)} \phi_2\left(Y_0 + i\right)$$

for $l \geq 1$, where C^j_k are the binomial coefficients and $\nabla^0_z f(z) = 1$. Note that the product in the last equation is equal to unity if the upper limit is less than the lower one.

In contrast to the case of kinematic resonances discussed in the previous section, the number of possible resonant interactions appearing in the Hamiltonian (8.29) is infinite. These interactions (*dynamic resonances*) can be classified as follows:

1. Principal (explicit) resonance, in which $k = 1, l = 0$, so that $\omega = \Omega$, corresponding to the interaction term that is explicitly present in the original Hamiltonian $\sim X_+ Y_-$.
2. Higher-order resonances, in which $\left(2l + k\right)\Omega = k\omega$, where $l, k \in N$ $(l \geq 1, k \geq 1)$, corresponding to the effective interactions $\sim X^k_+ Y^{2l+k}_-$, which can be further divided into
   - odd resonances: $k = 1,\ l = 1, 2, 3, \ldots$, so that $\omega = \left(2l + 1\right)\Omega$; in the case of the spin-spin interaction the third-order resonance is plotted in Figure 8.2;
   - even resonances: $k = 2, l = 1, 3, 5\ldots$, so that $\omega = 2m\Omega,\quad m = 1, 2, \ldots$;
   - fractional resonances: $\omega = \left(2l/k + 1\right)\Omega$, where k and l are coprime numbers (only those produce different resonances). The width of (k, l) resonance is of order $g\delta^{l+k-1}\varepsilon^{l+2(k-1)}$.

It is worth noting that in the vicinity of each resonance $\left(2l + k\right)\Omega = k\omega$, only the interaction term $\sim X^k_+ Y^{2l+k}_-$ survives. This means that if some resonance condition is satisfied, the system is in an approximate invariant subspace and there exists a corresponding approximate integral of motion, $N^{(k,l)}_{\text{eff}} = \left(2l + k\right)X_0 + kY_0$ $[N_{\text{eff}}, H_{\text{eff}}] \approx 0$, so that $[N, H] \approx 0$, where

$$N^{(kl)} = \left(2l + k\right)X_0 + kY_0 + 2k\delta\left(X_+ + X_-\right)\left(Y_+ + Y_-\right)$$
$$-\frac{2k^2\delta^2}{l\left(k + l\right)}\left[(k + l)\nabla_{X_0, -Y_0}\left[\phi_1\left(X_0\right)\phi_2(Y_0 + 1)\right] + l\nabla_{X_0, Y_0}\left[\phi_1\left(X_0\right)\phi_2(Y_0)\right]\right]$$

is obtained by applying the inverse set of transformations.

In the case when $X_\pm$ and/or $Y_\pm$ are nilpotent operators of qth order, all the powers beginning with $q + 1$ turn to zero, which obviously imposes a natural restriction on the series of possible resonances.

This simple example shows that even in one-channel systems, an infinite number of resonant interactions may arise if the total number of excitations is not preserved.

The appearance of such *dynamic* resonances in chaotic-like systems is expected from the point of view of classical dynamic systems [91]. Nevertheless, the quantum nature of interacting subsystems imposes certain restrictions on the possibility of survival of the dynamic resonances. We discuss such restrictions for two examples of interaction of an atomic system with quantum and classical fields.

8.3.1
Atom–Quantized Field Interaction

We consider a collection of A identical two-level atoms interacting with a single mode of a quantized field (the Dicke model) without RWA. The Hamiltonian that describes this system is

$$H = \Omega a^\dagger a + \omega S_z + g(aS_+ + a^\dagger S_-) + g(a^\dagger S_+ + aS_-) \tag{8.30}$$

and the condition $\omega, \Omega \gg g$ holds.

The following identifications should be done:

$$X_0 = S_z,\ X_\pm = S_\pm, \quad Y_0 = n, Y_+ = a^\dagger, \quad Y_- = a$$

so that $\phi_1(X_0) = (1 + A/2)A/2 - X_0^2 + X_0, \phi_2(Y_0) = Y_0$, lead to $\theta_{kl}\,(S_z, n) = 0,\ k \geq 3$ and the effective Hamiltonian (8.29) takes the form

$$\begin{aligned} H_{\text{eff}} \approx{}& \omega S_z + \Omega a^\dagger a + g\varepsilon\left(S_z^2 + \left(2a^\dagger a + 1\right) S_z - \left(1 + \frac{A}{2}\right)\frac{A}{2}\right) \\ &+ g\sum_{l=0}^{\infty} \frac{(-2\delta\varepsilon)^l}{l!}\left[a^{2l+1} S_+ + h.c.\right] - g\varepsilon \sum_{m=1}^{\infty} \frac{(4\delta\varepsilon)^{2m}}{(2m)!}\left[a^{4m} S_+^2 + h.c.\right] \end{aligned} \tag{8.31}$$

where $\varepsilon = g/\left(\Omega + \omega\right), \delta = g/2\Omega \ll 1$.

Note that the Hamiltonian (8.31) contains only the principal (explicit) resonance ($l = 0$), the even and the odd order interactions, but no fractional resonances. This happens because the structural functions for our subsystems are a first-degree polynomial of the photon number operators $a^\dagger a$ and a second-degree polynomial of the atomic population operator S_z.

The integrals of motion for the series of odd and even (exact) resonances are

$$\begin{aligned} N^{(kl)} ={}& (2l + k)S_z + ka^\dagger a + 2k\delta\left(S_+ + S_-\right)\left(a^\dagger + a\right) - \\ &- \frac{2k^2\delta^2}{l\left(l + k\right)}\left[\left(2l + k\right) S_z \left(2a^\dagger a + 1\right) - kS_z^2\right] + O\left(\delta^3\right) \end{aligned}$$

where $l = 2m + 1$ in $N^{(2l)}$, $m = 0, 1, 2, 3, \ldots$ and $k = 1, 2$ for even and odd resonances, respectively.

We recall that the higher resonances appear in the effective Hamiltonian (8.31) only under the condition $\omega \geq \Omega$. In the opposite case, $\omega \leq \Omega$, only the principal resonance survives and in the approximation (8.29) the whole effect of the counter-rotating terms reduces to the dynamic Stark shift, which has the same form as in the above case.

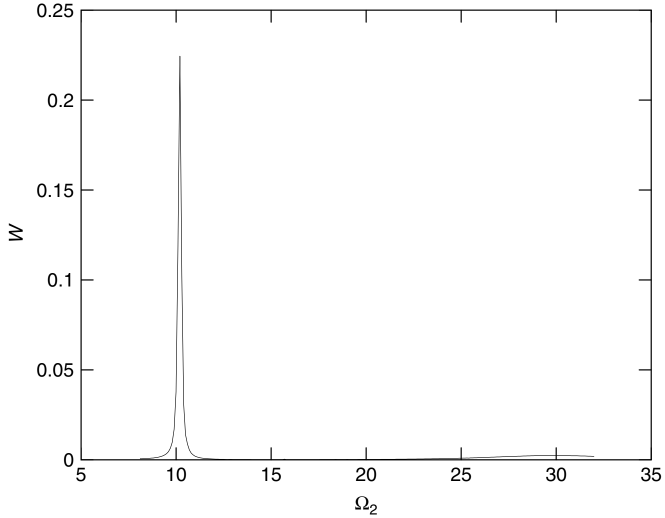

Figure 8.2 The third-order resonance in the spin–spin interaction $\omega_1 = 30$, $g = 1$. The temporal average transition probability between the spin states $W = \left\langle |_2\langle -3, 3|_1\langle 1, 1|U(t)|0, 1\rangle_1|0, 3\rangle_2|^2 \right\rangle_t$ against ω_2.

8.3.2
Atom–Classical Field Interaction

We proceed with the analysis of the interaction of atomic systems with classical fields without RWA. A generic interaction Hamiltonian has the form

$$H = \sum_{jj} \omega_j X_0^j + \sum_j g_j \left(X_j + X_j^\dagger \right) \cos(\Omega_j t + \vartheta_j) \tag{8.32}$$

and admits some explicit resonances, $\omega_j = \Omega_j$. The whole set of effective (implicit) interactions can be easily obtained in the same way as it was outlined above. To be able to use the effective Hamiltonian in the form (8.29) we first rewrite the Hamiltonian (8.32) in the Floquet form by making use of the phase operators $\{E_{0j}, E_j\}$, which are generators of the Euclidean algebra:

$$\left[E_{0j}, E_k\right] = -\delta_{jk} E_j, \quad \left[E_{0j}, E_k^\dagger\right] = \delta_{jk} E_j^\dagger, \quad \left[E_j, E_k^\dagger\right] = 0 \tag{8.33}$$

Each group of operators (labeled with the same index j) acts in a Hilbert space spanned by the eigenstates of the (Hermitian) operator E_{0j} :

$$E_{0j}|n\rangle_j = n|n\rangle_j, \quad n = \ldots - 1, 0, 1, \ldots \tag{8.34}$$

so that in the basis (8.34), the operators $E_j, E_j^\dagger$ act as raising–lowering operators:

$$E_j|n\rangle_j = |n-1\rangle_j, \quad E_j^\dagger|n\rangle_j = |n+1\rangle_j$$

The phase states $|\vartheta\rangle_j$ (which are the eigenstates of E_j, $E_j|\rangle_j = \mathrm{e}^{-\mathrm{i}\vartheta_j}|\vartheta\rangle_j$) are not normalized.

Now, let us consider the following time-independent Hamiltonian

$$H^{op} = \sum_j \omega_j h_j + \sum_{jj} \Omega_j E_{0j} + \sum_j g_j \left(X_j + X_j^\dagger\right)(E_j + E_j^\dagger) \tag{8.35}$$

It is easy to observe that the average value of the above Hamiltonian over the phase states in the rotating frame

$$|\vartheta(t)\rangle = \Pi_j \mathrm{e}^{-\mathrm{i}\Omega_j t E_{0j}}|\vartheta\rangle_j \tag{8.36}$$

$$|\vartheta\rangle_j = \lim_{N\to\infty} \frac{1}{\sqrt{2N+1}} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\vartheta_j n}|n\rangle_j$$

where $|n\rangle_j$ are the basis states (8.34), coincides with the Hamiltonian (8.32) because

$$\langle\vartheta(t)|E_{0j}|\vartheta(t)\rangle = 0, \quad \langle\vartheta(t)|E_j|\vartheta(t)\rangle = \mathrm{e}^{-\mathrm{i}(\Omega_j t + \vartheta_j)}$$

The Hamiltonian (8.35) is a Floquet form of the initial time-dependent Hamiltonian (8.32) (from now on we equate all the phases ϑ_j to zero). In the case of weak driven fields, $g \ll \omega, \Omega$, the Hamiltonian (8.35) can be represented in the form of an expansion over the principal resonances that can be observed in this system according to Equation 8.29.

In the single-channel case, the Dicke model in the classical field, we have the classical problem, first solved by Shirley [22] (see also Chapter 2) for a single two-level atom. Let us consider a collection of A two-level atoms in a linearly polarized electromagnetic field. The corresponding Hamiltonian has the form

$$H = \omega S_z + g(S_- + S_+)\cos\Omega t \tag{8.37}$$

where Ω is the classical field frequency. The Floquet form of Equation 8.37 is

$$H^{op} = \omega S_z + \Omega E_0 + g(S_- + S_+)(E + E^\dagger) \tag{8.38}$$

The following identifications

$$X_0 = S_z, \quad X_\pm = S_\pm, \quad Y_0 = E_0, \quad Y_+ = E^\dagger, \quad Y_- = E$$

so that

$$\phi_1(X_0) = (1 + A/2)A/2 - X_0^2 + X_0, \phi_2(Y_0) = E^\dagger E = EE^\dagger = I$$

lead to $\theta_{kl}(S_z, n) = 0$, $k \geq 2$, which immediately gives the effective Hamiltonian (8.29)

$$H_{\text{eff}}^{op} = \omega S_z + \Omega E_0 + 2g\varepsilon E^\dagger E S_z + g \sum_{l=0}^{\infty} \frac{(-2\delta\varepsilon)^l}{l!}\left(E^{2l+1}S_+ + h.c.\right) \tag{8.39}$$

Averaging the above Hamiltonian over the phase states we obtain an effective time-dependent Hamiltonian

$$H_{\text{eff}} = \omega S_z + 2g\varepsilon S_z + g\sum_{l=0}^{\infty}\frac{(-2\delta\varepsilon)^l}{l!}\left(e^{-i(2l+1)\Omega t}S_+ + h.c.\right)$$

which means that only odd resonances, $\Omega \approx \left(2l+1\right)\omega$, appear in this system. This result is different from the atom–quantum field interaction (and, surprisingly, cannot be obtained from the corresponding quantum Hamiltonian (8.31) by substituting the field operators by c-numbers) because, in this case, the structural function (8.22) for the Euclidean algebra (8.33) is a constant, that is, a zeroth-degree polynomial function.

8.3.3
Interaction of Atoms with the Quantum Field in the Presence of Classical Fields

In a separate, but very similar manner, the appearance of dynamic resonances in the process of interaction of a mode of quantum field with an atomic system driven by a classical field should be considered [95]. The Hamiltonian describing such a process has the form

$$H = \omega_1 S_z + \omega_2 a^\dagger a + g_1(aS_+ + a^\dagger S_-) + g_2(S_+ e^{-i\omega_0 t} + S_- e^{i\omega_0 t}) \tag{8.40}$$

where ω_1 and ω_2 are the atomic and field frequencies, and g_1 and g_2 are the coupling constants with the quantum and with the classical field, respectively. Usually, the classical field intensity is essentially larger than the interaction strength between the atomic system and the quantum field, so that the RWA can be applied. In a frame rotating at a frequency of the driving field, the above Hamiltonian acquires a time-independent form

$$H = \Delta_1 S_z + \Delta_2 a^\dagger a + g_1(aS_+ + a^\dagger S_-) + g_2(S_+ + S_-) \tag{8.41}$$

where the detuning parameters are defined as $\Delta_{1(2)} = \omega_{1(2)} - \omega_0$.

We observe that two kinds of resonant interactions are explicitly present in Equation 8.41: (i) the atomic system at resonance with the driving field, $\Delta_1 = 0$; (ii) the atomic system at resonance with the quantum field, $\Delta_2 = \Delta_1$, that encloses the particular case of the "full" resonance: $\Delta_2 = \Delta_1 = 0$. Nevertheless, as it is clear now, an infinite number of higher-order resonances are also present. All these implicit resonant transitions are, of course, far away from the above-mentioned principal resonances. This means that the following conditions hold:

$$g_2 \ll \Delta_1, \quad g_1 \ll \omega_1 - \omega_2 \tag{8.42}$$

Thus, the terms describing principal transitions can be adiabatically eliminated from the Hamiltonian (8.41) according to the standard procedure. To eliminate the terms $g_2(S_+ + S_-)$ and $g_1(aS_+ + a^\dagger S_-)$ we apply to the Hamiltonian 8.41 two consecutive transformations

$$U_1 = \exp\left[\varepsilon(S_+ - S_-)\right], \quad \varepsilon = \frac{g_2}{\Delta_1} \ll 1 \tag{8.43}$$

$$U_2 = \exp\left[\epsilon(aS_+ - a^\dagger S_-)\right], \quad \epsilon = \frac{g_1}{\omega_1 - \omega_2} \ll 1 \tag{8.44}$$

where ε and ϵ are small parameters as follows from Equation 8.42. The transformed Hamiltonian

$$H_2 = U_2 U_1 H U_1^\dagger U_2^\dagger$$

contains numerous terms describing different physical processes. These terms can be classified into three groups: (i) *diagonal terms,* which depend exclusively on S_z and $a^\dagger a$, and are usually called *dynamical Stark shifts*; (ii) *nondiagonal resonant terms,* which, under certain conditions on the transition frequencies, describe resonant transitions or, in other words, those that become time independent in the rotating frame if certain relations between Δ_1 and Δ_2 are fulfilled; (iii) *nondiagonal terms* that would never represent resonant interactions, and thus, could be adiabatically eliminated from $\tilde{H}$ in a standard way.

Retaining only diagonal and nondiagonal resonant terms in the Hamiltonian, we arrive at the following generic form (up to a constant energy shift):

$$\begin{aligned} H_2 = {} & \left(\Delta_1 + 2g_2\varepsilon\right) S_z + \Delta_2 a^\dagger a + g_1\epsilon\left((2a^\dagger a + 1)S_z - S_z^2 + O(\varepsilon^2)\right) \\ & + 2g_1\varepsilon S_z\left(a + a^\dagger\right) - g_1\epsilon\varepsilon\sum_{n,k}\left[h_{nk}(\varepsilon, \epsilon, S_z, a^\dagger a)a^n S_+^k + h.c.\right] \\ & - g_1\varepsilon^2\sum_{n,k}\left(f_{nk}(\varepsilon, \epsilon, S_z, a^\dagger a)a^n S_-^k + h.c.\right) \end{aligned} \tag{8.45}$$

where h_{nk} and f_{nk} are functions depending on diagonal operators in the following form:

$$f_{nk}, h_{nk} \sim \frac{\xi^m}{m!}\sum_q \frac{\xi^{2q+P_1(n,k)}}{(2q + P_1(n,k))!}P_q(a^\dagger a, S_z), \quad m \geq \max\left(n, k\right)$$

where $P_1(n,k)$ is a first-order polynomial in n and k, $P_q(a^\dagger a, S_z)$ is a qth order polynomial in diagonal operators $a^\dagger a$ and S_z whose coefficients grow no faster than $(q+1)!$, and ξ is a small parameter in such a way that $\xi^s \sim \varepsilon^l \epsilon^{s-l}$.

The Hamiltonian (8.45) contains several terms that describe resonant transitions under certain conditions on the frequencies. The most important term, $\sim \varepsilon S_z\left(a + a^\dagger\right)$, describes an intensity-dependent field excitation that becomes resonant if the external field is in resonance with the quantum field, $\omega_2 \approx \omega_0$ ($\Delta_2 \approx 0$). Obviously, all the other nondiagonal terms in Equation 8.45 are out of resonance and do not essentially contribute to the system dynamics, i.e. under the condition

$\omega_2 \approx \omega_0$, the effective Hamiltonian describing the system evolution has the form (in the rotating frame)

$$\tilde{H} \approx \left(\Delta_1 + 2g_2\varepsilon\right) S_z + g_1\epsilon \left((2a^\dagger a + 1)S_z - S_z^2\right) + 2g_1\varepsilon S_z \left(a + a^\dagger\right)$$

If instead, the classical and quantum fields are out of resonance so that $\Delta_2 \gg g_1\varepsilon$, the term $2g_1\varepsilon S_z \left(a + a^\dagger\right)$ can be eliminated in a similar way as has been done before, by applying the following transformation:

$$U_3 = \exp\left[2\delta\varepsilon S_z \left(a^\dagger - a\right)\right], \quad \delta = \frac{g_1}{\Delta_2}$$

where δ is not necessarily a small parameter, but rather the condition $\delta\varepsilon \ll 1$ holds. Then, conserving only the leading order in the coefficients, we obtain that the effective Hamiltonian $H_{\text{eff}} = U_3 H_2 U_3^\dagger$ takes the form

$$\begin{aligned} H_{\text{eff}} \approx {} & \left(\Delta_1 + 2g_2\varepsilon\right) S_z + \Delta_2 a^\dagger a + g_1\epsilon \left((2a^\dagger a + 1)S_z - S_z^2\right) \\ & - 2g_1\epsilon\varepsilon \sum_{n=0}^{\infty} \frac{(-2\delta\varepsilon)^n}{n!} \left[a^{n+2}S_+ + h.c.\right] \\ & - g_1\varepsilon^2 \sum_{n=0}^{\infty} \frac{(-2\delta\varepsilon)^n}{n!} \left[a^{n+1}S_- + h.c.\right] \\ & - 2g_1\epsilon^2\varepsilon \sum_{n=0}^{\infty} \frac{(-4\delta\varepsilon)^{2n}}{(2n)!} \left[a^{2n+1}S_+^2 + h.c.\right] \end{aligned} \tag{8.46}$$

Note that not all the transitions that formally appear in Equation 8.45 are present in the effective Hamiltonian (8.46) because of the specific form of the field and atomic commutation relations.

Three different kinds of resonances are present in the effective Hamiltonian (8.46):

1. *Individual rotating-type resonances*: These do not depend on the number of atoms; we refer to this as satisfying the condition $(n+2)\omega_2 = (n+1)\omega_0 + \omega_1 + \zeta_1^{(n)}$; $n = 0, 1, \ldots$, where $\zeta_1^{(n)} = O(\varepsilon)$ is a small frequency shift. In the vicinity of each rotating-type resonance, the excitation-number operator $\hat{N}_1^{(n)} = (n+2)S_z + a^\dagger a$ approximately commutes with the Hamiltonian, and the corresponding number of excitations is approximately preserved. These resonances can be found even for a single two-level atom.
2. *Antirotating type resonances:* These are similar to the Previous type, but are of the antirotating form, and satisfy the condition $(n+1)\omega_2 = (n+2)\omega_0 - \omega_1 + \zeta_2^{(n)}$; $n = 0, 1, \ldots$, where $\zeta_2^{(n)} = O(\varepsilon)$ is another small frequency shift. The corresponding approximate integral of motion under this condition is $N_2^{(n)} = (n+1)S_z - a^\dagger a$.
3. *Collective rotating-type resonances:* The condition satisfied is $(2n+1)\omega_2 = (2n-1)\omega_0 + 2\omega_1 + \zeta_3^{(n)}$; $n = 0, 1, \ldots$, where $\zeta_3^{(n)} = O(\varepsilon)$ is the frequency shift.

In the vicinity of each resonance of this type, the excitation-number operator $N_3^{(n)} = (2n+1)S_z + 2a^\dagger a$ approximately commutes with the Hamiltonian and the corresponding number of excitations is approximately preserved. These resonances have a collective nature and are obviously absent in the case of a single atom.

It is clear that to each resonance corresponds an approximately invariant subspace, and its dimension depends on the number of photons and/or atoms. The frequency shifts $\zeta_j^{(n)}$ are different for each invariant subspace, i.e. they depend on the value of the corresponding integral of motion, $\zeta_j^{(n)} = \zeta(N_j^{(n)})$.

As a first example, we write the effective Hamiltonian – in its corresponding rotating frame – governing the evolution of a single two-level atom in the vicinity of the second individual rotating-type resonance:

$$H_{\text{eff}}^{(2R)} \approx \left(2g_2\varepsilon + g_1\epsilon(2N_1^{(0)}+1) - \zeta_1^{(0)}\right) S_z - 2g_1\epsilon\varepsilon\left[a^2 S_+ + a^{\dagger 2} S_-\right] \tag{8.47}$$

where the resonance condition is $2\omega_2 = \omega_0 + \omega_1 + \zeta_1^{(0)}$. Similarly, for the third individual rotating-type resonance and a single atom, we have

$$H_{\text{eff}}^{(3R)} \approx \left(2g_2\varepsilon + g_1\epsilon(2N_1^{(1)}+1) - \zeta_1^{(1)}\right) S_z + 4g_1\epsilon\delta\varepsilon^2\left[a^3 S_+ + h.c.\right]$$

In the case of a first antirotating type, $H_{\text{eff}}^{(1A)}$, resonance, we get

$$H_{\text{eff}}^{(1A)} \approx \left(2g_2\varepsilon + g_1\epsilon(1-2N_2^{(0)}) + \zeta_2^{(0)}\right) S_z - g_1\varepsilon^2\left(a^\dagger S_+ + aS_-\right) \tag{8.48}$$

where $\omega_2 = 2\omega_0 - \omega_1 + \zeta_2^{(0)}$. As is noticed, this resonance is of the same order of magnitude in the perturbation parameter as is the second individual rotating-type resonance. Hence, they evolve at the same timescale.

In Figure 8.3a, we plot the temporal average of the transition probability $|\langle 0;2|U_{\text{eff}}^{(2R)}|1;0\rangle|^2$ for the case of the second resonance, as a function of the frequency of the driving field, ω_0, in units of the coupling constant g. In this case, $\omega_1/g = 50$ and $\omega_2/g = 35$. It can be observed that the center of the resonance is at $\omega_0/g = 19.55$; this value satisfies the condition $2\omega_2 \approx \omega_0 + \omega_1$. Here the state $|n;m\rangle$denotes bare atomic state $|n\rangle$ and field number state $|m\rangle$. The same notation is used in the figures that follow. All the figures have been obtained from the exact numerical evolution given by Equation 8.41. Several effective Rabi oscillations have been included; this explicitly shows that there are no other terms in the series expansion, affecting the evolution under specific resonance conditions. In Figure 8.3b , we plot the temporal evolution of the same transition probability. This figure shows that, with high probability, all the initial population is transferred to the final state.

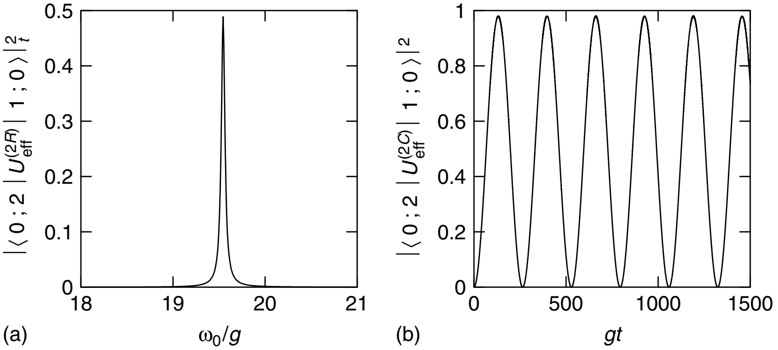

Figure 8.3 (a) Temporal average and (b) temporal evolution of transition probability, $|\langle 0;2|U_{\text{eff}}^{(2R)}|1;0\rangle|^2$, in the case of a second resonance, $2\omega_2 = \omega_0 + \omega_1 + \zeta_1^{(0)}$, as a function of driving frequency. The resonance occurs for $\omega_0/g = 19.55$, which corresponds to a frequency shift $\zeta_1^{(0)}/g = 0.45$. $\omega_1/g = 50$, $\omega_2/g = 35$.

In Figure 8.4 we plot the transition probability $|\langle 0;3|U_{\text{eff}}^{(3R)}|1;0\rangle|^2$. Here two important aspects should be noted: (i) the resonance is much narrower than the second resonance, by a factor of order ~100; (ii) this implies that the population transfer occurs on a longer temporal scale. However, under both conditions, the population transfer occurs with a probability close to one; this means that each nonlinearity is precisely selected. The parameters in all the figures were chosen exclusively for illustrative reasons, in such a way so as to clearly show the emergence of the effective resonances for reasonable computational times.

In Figure 8.5 we plot the transition probability $|\langle 1;1|U_{\text{eff}}^{(1A)}|0;0\rangle|^2$ for the conditions of the first antirotating type resonance. It can be noticed that the width of this

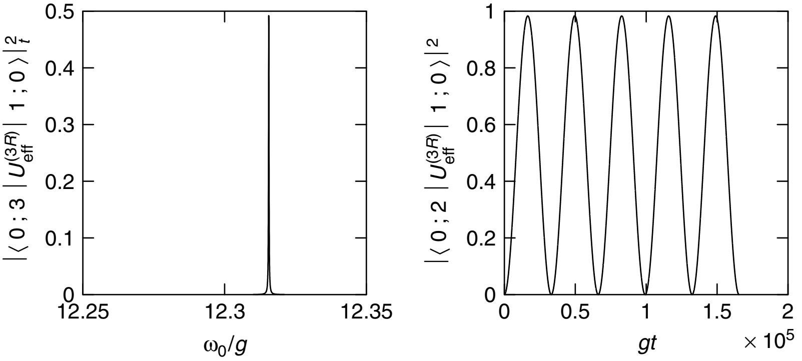

Figure 8.4 Transition probability $|\langle 0;3|U_{\text{eff}}^{(3R)}|1;0\rangle|^2$ in the case of a third individual type resonance, $3\omega_2 = 2\omega_0 + \omega_1 + \zeta_1^{(1)}$. In this case, $\omega_1/g = 50$ and $\omega_2/g = 25$. The center of the resonance is at $\omega_0/g = 12.3156$.

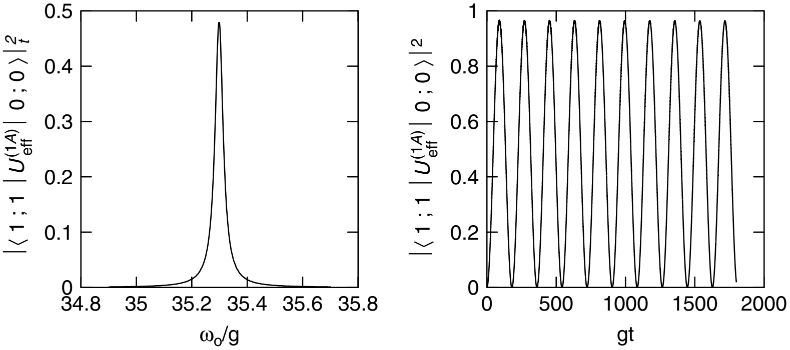

Figure 8.5 First antirotating type resonance, $\omega_2 = 2\omega_0 - \omega_1 + \zeta_2$. In this case, $\omega_1/g = 50$ and $\omega_2/g = 20$. The center of the resonance is at $\omega_0/g = 35.299$.

resonance is of the same order as for the second individual rotating-type resonance, i.e. they evolve in the same timescale.

As a second, less evident example, we obtain from Equation 8.46 an effective Hamiltonian describing a collective excitation of atoms by a single photon:

$$H_{\mathrm{eff}}^{(2C)} \approx \left(2g_2\varepsilon + g_1\epsilon(N_3^{(0)} - S_z + 1) - \frac{\zeta_3^{(0)}}{2}\right) S_z \\ - 2g_1\epsilon^2\varepsilon\left[aS_+^2 + h.c.\right] \tag{8.49}$$

where the resonance condition is $\omega_2 + \omega_0 = 2\omega_1 + \zeta_3^{(0)}$. It is worth noting that, in this case, the dynamical Stark shift term cannot, in general, be dynamically compensated by the atomic frequency shift. This may lead, in particular, to oscillations in the temporal evolutions of the transition probabilities. In Figure 8.6

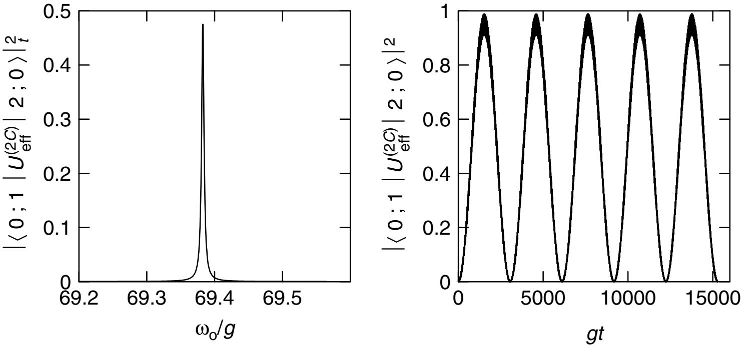

Figure 8.6 Transition probability $|\langle 0;1|U_{\mathrm{eff}}^{(2C)}|2;0\rangle|^2$ in the case of a two-atom resonance, $2\omega_1 = \omega_0 + \omega_2 + \zeta_2$. In this case, $\omega_1/g = 50$ and $\omega_2/g = 30$. The center of the resonance occurs for $\omega_0 = 69.383$.

we plot the transition probability $|\langle 0;1|U_{\text{eff}}^{(2C)}|2;0\rangle|^2$. In this case, the resonance is narrower than the second individual type resonance by a factor of about 10, so the exchange of populations occurs on a different timescale.

Thus, quantum systems not preserving the total number of excitations do not admit a simple classification of possible resonant transitions. The explicit form of the effective Hamiltonian is needed to specify the allowed resonances. Such nonlinear (on the generators of some Lie algebra) Hamiltonians can be represented as a series of operators describing all the possible transitions, which might become resonant under specific relations between frequencies of interacting subsystems. The structure of the effective Hamiltonian essentially depends on the algebraic structure of interacting subsystems (polynomials $P_1(X_0)$, $P_2(Y_0)$). In particular, it is reflected in the types of resonances that are allowed for a given system.

8.4
Dynamics of Slow and Fast Interacting Subsystems

Now we turn to the intermediate regime, $\omega_n \lesssim g_{jk} \ll \Omega_m$ (or $\Omega_m \lesssim g_{jk} \ll \omega_n$), when one can separate the systems into fast and slow interacting subsystems. It is well known that, in this case, we can adiabatically eliminate the fast subsystems, the effect of which on the slow system dynamics is taking into account as an effective potential (the Born–Oppenheimer approximation). In this section, we analyze the effective Hamiltonians describing evolution of a generic quantum system X interacting with a quantum system Y (one quantum channel) when the total excitation number is not preserved in the case where the characteristic frequency of the system X is essentially lower than the corresponding frequency of the system Y, $\omega_X \ll \omega_Y$, and the interaction constant g satisfies the strong coupling condition: $\omega_X \lesssim g \ll \omega_Y$. We show that, depending on the type of interaction and the nature of the quantum systems, different physical situations take place, but generically such effective Hamiltonians describe QPTs in the *slow system*. It is important to stress that the RWA cannot be directly applied because the contributions of the antirotating and rotating terms in the Hamiltonian are of the same order.

Let us consider the same generic Hamiltonian as in Section 8.3 describing an interaction between two quantum systems:

$$H = \omega_1 X_0 + \omega_2 Y_0 + g\,(X_+ + X_-)\,(Y_+ + Y_-) \tag{8.50}$$

where X_0 and Y_0 are the free Hamiltonians of the X and Y systems respectively, and such that $\omega_1 \lesssim g \ll \omega_2$. Applying the Lie transformation method, we can adiabatically remove all the terms that contain the fast system's transition operators, $Y_\pm$. In particular, the counter-rotating term $X_+Y_+ + X_-Y_-$ and the rotating term $X_+Y_- + X_-Y_+$ can be eliminated from the Hamiltonian (8.50) by a subsequent application of the following transformations:

$$U_1 = \exp\left[\varepsilon\,(X_+Y_+ - X_-Y_-)\right] \tag{8.51a}$$

$$U_2 = \exp\left[\epsilon\,(X_+Y_- - X_+Y_-)\right] \tag{8.51b}$$

where the small parameters, ε and ϵ, are defined by

$$\varepsilon = \frac{g}{\omega_2 + \omega_1} \ll 1, \quad \epsilon = \frac{g}{\omega_2 - \omega_1} \ll 1 \tag{8.52}$$

The transformations (8.51a) and (8.51b) generate different kinds of terms, such as, $X_\pm^n Y_\pm^k + h.c.$, $X_\pm^n Y_\mp^k + h.c.$, $Y_\pm^n + h.c.$, and $X_\pm^n + h.c.$ with coefficients depending on X_0 and Y_0. Under the condition $\omega_1, g \ll \omega_2$, all the rapidly oscillating terms, i.e. those containing powers of $Y_\pm$, can be removed by applying transformations similar to Equation 8.51a and 8.51b, with properly chosen parameters. Nevertheless, the powers of $X_\pm$ cannot be eliminated now, which is the main difference from the procedure applied in Section 8.3.

Here, we restrict ourselves to terms up to the third order in δ, which are sufficient to analyze the phenomena of our interest. Then, the transformation required to eliminate the terms that contain the transition operators $Y_\pm$, which appear after applying Equation 8.51a and 8.51b, has the form

$$U_3 = \exp\left[\frac{\delta\varepsilon}{2}\left(Y_+^2 - Y_-^2\right)\nabla_x\phi_1(X_0)\right] \tag{8.53}$$

The result can be expressed as a power series of the single parameter $\delta = g/\omega_2 \ll 1$.

Taking into account the above-mentioned considerations, keeping only terms up to third order in δ and disregarding small corrections to the effective transition frequencies, we arrive at the following effective Hamiltonian, diagonal (in our approximation) on the operators of the Y system:

$$\begin{aligned} H_{\text{eff}} = {} & \omega_1 X_0 + \omega_2 Y_0 - 2\omega_1\delta^2\nabla_{x,-y}\left(\phi_1\left(X_0\right)\phi_2(Y_0)\right) + g\delta\nabla_y\phi_2(Y_0) \\ & (X_+ + X_-)^2 + \frac{1}{2}g\delta^3\nabla_y\left(\phi_2(Y_0)\nabla_y^2\phi_2(Y_0 - 1)\right)(X_+ + X_-)^4 \end{aligned} \tag{8.54}$$

The generalized displacement operators are defined as

$$\nabla_{mX_0,nY_0} f(X_0, Y_0) = f(X_0, Y_0) - f(X_0 + m, Y_0 + n)$$

where m and n integers.

Because the effective Hamiltonian (8.54) is diagonal on the operators of the Y *fast system*, we may project it onto a minimal energy eigenstate of the Y system, $|\psi_0\rangle_Y$, substituting Y_0 by its eigenvalue y_0: $Y_0|\psi_0\rangle_Y = y_0|\psi_0\rangle_Y$, where the parameter y_0 is usually directly related to the dimension of the system Y.

The first-order effect comes from the term $\sim (X_+ + X_-)^2$, while the term $\sim (X_+ + X_-)^4$ defines a *fine structure* of the *effective potential*, obtained after projecting the effective Hamiltonian (8.54) onto the state $|\psi_0\rangle_Y$.

It is important to stress that, although δ is a small parameter, the effect of the terms $\sim \delta^n$, $n \geq 1$, could be, in principle, comparable with the main diagonal term $\omega_1 X_0$, especially if the algebra of X operators describes a *big subsystem*, i.e., large spin or big photon number. In this case, nontrivial effects such as QPTs may occur.

The QPTs are a common feature of nonlinear quantum systems. Such transitions occur at zero temperature and are associated with an abrupt change in the ground state structure. QPTs are related to singularities in the energy spectrum and, at the critical points defining QPTs, the ground state energy is a nonanalytic function of the system's parameters [96]. Qualitatively, for a wide class of quantum systems, several important properties of QPTs can be studied in the thermodynamic [97] and semiclassical limits [98, 99]. Then, QPTs can be analyzed in terms of a classical effective potential energy surface [100]. In this language, QPTs are related to the appearance of a new classical separatrix when the coupling parameters acquire certain values. According to the standard semiclassical quantization scheme and the correspondence principle, the energy density is proportional to the classical period of motion, diverging on the separatrix, which explains a high density of quantum states at the critical points.

To detect such an effect of QPTs we proceed with analysis of the effective Hamiltonian (8.54) of the Dicke model, focusing on the possible bifurcation of the ground state. The simplest way to study the critical properties of the effective Hamiltonian (8.54) consists in taking the thermodynamic limit, when, after an appropriate rescaling of the interaction constants (see [101]) we let the dimension of the fast system tend to infinity. The rescaled Hamiltonian has the form

$$H = \omega_1 \hat{n} + \omega_2 S_z + \frac{g}{\sqrt{A}} (S_+ + S_-) \left(a^\dagger + a\right) \tag{8.55}$$

where $\hat{n} = a^\dagger a$ and $S_{z,\pm}$ are generators of the $(A+1)$-dimensional representation of the $su(2)$ algebra.

8.4.1
Effective Field Dynamics

First, we suppose that the atoms form a *fast subsystem* so that,

$$X_0 = \hat{n}, \quad X_+ = a^\dagger, \quad X_- = a, \quad Y_0 = S_z, \quad Y_\pm = S_\pm$$

and hence, $\phi_2(Y_0) = C_2 - S_z^2 + S_z$ and $\phi_1(X_0) = \hat{n}$, where $C_2 = A/2(A/2+1)$ is the eigenvalue of the Casimir operator of the $su(2)$ algebra (integral of motion corresponding to the atomic subsystem).

Projecting the effective Hamiltonian onto the minimum energy state of the atomic system $|0\rangle_{\text{at}}$, so that $y_0 = -A/2$, we obtain the following effective Hamiltonian for the field mode:

$$H_{\text{eff}} = \tilde{\omega}_1 \hat{n} - g\delta \left(a + a^\dagger\right)^2 + gA^{-1}\delta^3 \left(a + a^\dagger\right)^4 - \frac{A}{2}\omega_2 \tag{8.56}$$

where $\tilde{\omega}_1 = \omega_1(1 - 2\delta^2)$.

Rewriting Equation 8.56 in terms of position and momentum operators

$$H_{\text{eff}} = \frac{\tilde{\omega}_1}{2}(p^2 + x^2) - 2g\delta x^2 + 4gA^{-1}\delta^3 x^4 - \frac{A}{2}\omega_2$$

we immediately detect that the QPT, in this case, is related to the bifurcation of the effective potential $U(x) = (\tilde{\omega}_1/2 - 2g\delta)x^2 + 4gA^{-1}\delta^3 x^4$ (up to a constant shift) from a single minimum at $x = 0$ for $4g\delta < \tilde{\omega}_1$ to a symmetric double-well structure at $4g\delta > \tilde{\omega}_1$, with minima at $x_* = \pm\sqrt{\Delta A/(16\delta^3)}$, where $\Delta = 4\delta - \tilde{\omega}_1/g > 0$. We stress that the above effective potential describes the system behavior only in the weak excitation limit, $\Delta \ll \omega_1/g$.

The physical effect associated with this QPT consists of a spontaneous generation of photons in the field mode. In some sense, the virtual photons, always presented in the Dicke model (8.55), are *condensed* into the real photons after crossing the critical point $4g\delta = \tilde{\omega}_1$.

We can easily evaluate, in the leading order on Δ, the average number of photons created in the field (normalized by the number of atoms) after passing the phase transition point (superradiant phase) [97]

$$\frac{\overline{n}}{A} \approx \frac{1}{2A}\langle \hat{p}^2 + \hat{q}^2\rangle_f \approx \begin{cases} 0, & 4g\delta \leq \tilde{\omega}_1 \\ \frac{x_*^2}{2A} = \frac{\Delta}{(32\delta^3)} \approx \frac{\Delta\omega_2^2}{(8g\omega_1)}, & 4g\delta \geq \tilde{\omega}_1 \end{cases}$$

where $\langle\rangle_f$ means averaging over the field state, and $\hat{q} = (a + a^\dagger)/\sqrt{2}$, $\hat{p} = (a - a^\dagger)/i\sqrt{2}$; this result coincides with the one obtained in the thermodynamic limit taking this approximation.

8.4.2
Effective Atomic Dynamics

In the opposite case, when the atoms form a *slow subsystem* we have

$$X_0 = S_z, \quad X_\pm = S_\pm, \quad Y_0 = \hat{n}, \quad Y_+ = a^\dagger, \quad Y_- = a$$

Projecting the effective Hamiltonian onto the minimum energy state of the field mode $|0\rangle_f$, so that $y_0 = 0$, the effective Hamiltonian acquires the form

$$H_{\text{eff}} = \tilde{\omega}_1 S_z - 4\frac{g\delta}{A}S_x^2 + 2\frac{\omega_1\delta^2}{A}S_z^2 \tag{8.57}$$

where $\tilde{\omega}_1 = \omega_1 - 2\omega_2\delta^2/A$.

For our analysis, it is convenient to perform a $\pi/2$ rotation in Equation 8.57 around the y axis (this avoids the coincidence of the physically important South pole of the sphere with the singular point), transforming the Hamiltonian (8.57) into

$$\tilde{H}_{\text{eff}} = -\tilde{\omega}_1 S_x - 4\frac{g\delta}{A}S_z^2 + 2\frac{\omega_1\delta^2}{A}S_x^2 \tag{8.58}$$

In the semiclassical limit, we may replace the atomic operators by the corresponding classical vectors over the two-dimensional sphere, i.e.

$$S_z \to \frac{A}{2}\cos\theta, \quad S_x \to \frac{A}{2}\sin\theta\cos\phi, \quad S_y \to \frac{A}{2}\sin\phi\sin\theta$$

and thus rewrite the effective Hamiltonian (8.58) as a classical Hamiltonian function

$$H_{\mathrm{cl}} = -\frac{A}{2}(\tilde{\omega}_1 \cos\phi \sin\theta + 2g\delta \cos^2\theta \\ - \omega_1 \delta^2 \cos^2\phi \sin^2\theta \tag{8.59}$$

The first two terms in the above expression describe the semiclassical limit of the Lipkin–Meshkov–Glick model [99] and determine the critical point of the QPT, $\xi = 4g\delta/\tilde{\omega}_1 = 1$, which again is related to the bifurcation of the ground state: a single minimum at $\sin\phi = 0$, $\cos\theta_* = 0$ splits into two minima at $\sin\phi = 0$, $\cos\theta_{**} = \pm\sqrt{1-\xi^{-2}}$, for $\xi > 1$. It is worth noting that the global minimum of H_{cl} at $\xi < 1$ converts into a local maximum for $\xi > 1$, so that [99],

$$H_{\mathrm{cl}}(\theta_{**}) \approx -\frac{A}{4}(\xi + \xi^{-1}) < H_{\mathrm{cl}}(\theta_*) = -\frac{A}{2}$$

This means that the atoms, initially prepared at the minimum of the Hamiltonian function, spontaneously change their ground state energy at some value of the system's parameters. Classically, this implies appearance of a separatrix, which leads to the discontinuity on the energy density spectrum in the semiclassical limit. It is also worth noting that there is a loss of rotational symmetry in this process: the new ground state is obviously not invariant under rotations around the x axis, while the initial ground state is clearly invariant under x rotations.

It is easy to see that the last term in Equation 8.59 is of lower order in the parameter δ and can be neglected in the first approximation for a description of the QPT at $\xi = 1$.

8.5
Problems

8.1 Prove that there are no additional resonances in the Hamiltonian describing linear field–field interaction,

$$H = \omega_1 a^\dagger a + \omega_2 b^\dagger b + g\left(a^\dagger + a\right)\left(b^\dagger + b\right)$$

8.2 The Hamiltonian describing spin–spin (dipole–dipole) interaction has the following generic form:

$$H = \omega_1 S_{1z} + \omega_2 S_{2z} + g\left(S_{1+} + S_{1-}\right)\left(S_{2+} + S_{2-}\right), \quad g \ll \omega_{1,2}$$

where $S_{1z,\pm}$ and $S_{2z,\pm}$ are generators of $2S_1 + 1$ and $2S_2 + 1$ dimensional representations of the $su(2)$ algebra correspondingly. Prove that in addition to the odd $\omega_1 = (2m+1)\,\omega_2$ and even $\omega_1 = 2m\omega_2, m = 1, 2, \ldots$. Resonances, the first series of fractional resonances appears with $\omega_1 = \left(2l/3 + 1\right)\omega_2, l \neq 3N$, $N = 1, 2, \ldots$.

9
Models with Dissipation

9.1
Dissipation and Pumping of the Quantum Field

Let us consider a mode of the quantum field pumped by a classical force inside a cavity with losses (that is, the field dissipates because of interaction with the cavity walls). The equation that describes the evolution of the density matrix for the field is of the form

$$\partial_t \rho = -\mathrm{i}\,[H, \rho] + L\rho \tag{9.1}$$

where the field Hamiltonian H with (harmonic) pumping is

$$H = \omega_f a^\dagger a + \Omega\left(a\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{c}} t} + a^\dagger \mathrm{e}^{\mathrm{i}\omega_{\mathrm{c}} t}\right) \tag{9.2}$$

where Ω is the amplitude of the pumping (classical) field of frequency ω_{c} and a ($a^\dagger$) are field operators. We describe the dissipation by the Lindblad-like term [102],

$$L\rho = \frac{\gamma}{2}\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right) \tag{9.3}$$

which corresponds to the cavity without thermal photons. In the interaction frame, the Hamiltonian (9.2) takes on the form

$$H_I = \delta\, a^\dagger a + \Omega(a + a^\dagger) \tag{9.4}$$

where $\delta = \omega_f - \omega_{\mathrm{c}}$ is the shift between the field frequency and that of the pumping field. The Hamiltonian (9.4) is linear and, therefore, the master equation (Equation 9.1) can be solved by the dynamic symmetry method. According to this method, we introduce the following notation for the operators that appear in the master equation:

$$\begin{gathered} N_-\rho = a\rho a^\dagger, \quad N_l\rho = a^\dagger a\rho, \quad N_r\rho = \rho a^\dagger a \\ A_l\rho = a\rho, \quad A_{+l}\rho = a^\dagger\rho, \quad A_r\rho = \rho a, \quad A_{+r}\rho = \rho a^\dagger \end{gathered} \tag{9.5}$$

Clearly, all the operators that act from the of the density matrix commute with all the operators that are applied from the left to the right. The operators N_l, N_r, and

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N_- form a solvable subalgebra:

$$[N_l, N_-] = [N_r, N_-] = -N_- \,, \quad [N_l, N_r] = 0$$

$A_r\,, A_{+r}\,, 1$ and $A_l\,, A_{+l}\,, 1$ are two Heisenberg–Weyl subalgebras. The remaining commutators between the operators (9.5) are

$$\begin{aligned}
&[N_l, A_l] = -A_l\,, \quad [N_l, A_{+l}] = A_{+l}, \\
&[N_-, A_l] = 0\,, \quad [N_-, A_{+l}] = A_{+r}, \\
&[N_r, A_r] = A_r\,, \ [N_r, A_{+r}] = -A_{+r}, \\
&[N_-, A_r] = A_l\,, \ [N_-, A_{+r}] = 0
\end{aligned} \tag{9.6}$$

The master equation has a linear form in terms of the operators (9.5):

$$\partial_t \rho = [\gamma N_- + v N_l + v^* N_r + \mathrm{i}\Omega\,(A_r + A_{+r} - A_l - A_{+l})]\,\rho \tag{9.7}$$

where

$$v = -\mathrm{i}\delta - \frac{\gamma}{2}$$

and, therefore, its solution can be represented in the form [103, 104]

$$\rho(t) = \mathrm{e}^{\mu}\mathrm{e}^{v^* t N_r}\mathrm{e}^{v t N_l}\mathrm{e}^{f N_-}\mathrm{e}^{b^* A_{+l}}\mathrm{e}^{c A_l}\mathrm{e}^{b A_r}\mathrm{e}^{c^* A_{+r}}\,\rho(0) \tag{9.8}$$

ν where the equations to determine the parameters $\mu(t), f(t), b(t), c(t)$ are found by substituting Equation 9.8 into Equation 9.7,

$$\begin{aligned}
\dot{\mu} &= c^*\dot{b} + \dot{c}b^*, \quad \dot{f} = \gamma \mathrm{e}^{-\gamma t}, \\
\dot{b}f + \dot{c} &= -\mathrm{i}\Omega \mathrm{e}^{v^* t}, \quad \dot{b} = \mathrm{i}\Omega \mathrm{e}^{-v^* t}, \\
\mu(0) &= f(0) = b(0) = c(0) = 0
\end{aligned} \tag{9.9}$$

where $\rho(0)$ is the initial density matrix. The solution of the system (9.9) is

$$\begin{aligned}
\frac{\mu(t)}{\Omega^2} &= -\frac{1}{|v|^2}|\mathrm{e}^{-vt} - 1|^2, \\
b(t) &= -c(t) = \mathrm{i}\frac{\Omega}{v^*}\big(\mathrm{e}^{-v^* t} - 1\big), \\
f(t) &= 1 - \mathrm{e}^{-\gamma t}
\end{aligned} \tag{9.10}$$

Using the solution in the form (9.8), we find the evolution of a coherent initial state $\rho_f(0) = |\alpha\rangle\langle\alpha|$, obtaining the well-known result that the initial coherent state remains coherent and the density matrix of the field is of the form

$$\rho(t) = |z(t)\mathrm{e}^{-\mathrm{i}\omega_c t}\rangle\langle z(t)\mathrm{e}^{-\mathrm{i}\omega_c t}|$$

where

$$z(t) = (\alpha + b^*)\mathrm{e}^{vt}$$

and the parameter $b(t)$ is defined in Equation 9.10. In the stationary state ($t \to \infty$), the field density matrix takes the following form:

$$\rho^{ss} = |z^{ss}e^{-i\omega_c t}\rangle\langle z^{ss}e^{-i\omega_c t}|$$

where $z^{ss} = i\Omega/v$. The average number of photons in the stationary state is

$$\langle\hat{n}\rangle = \mathrm{Tr}\left(a^\dagger a\,\rho^{ss}\right) = \frac{\Omega^2}{|v|^2}$$

Note that this property of the coherence being conserved is maintained only when the initial state is a single coherent state. If, for example, the initial state of the field is a superposition of coherent states, the coherence among these states is rapidly lost and the whole system evolves into an incoherent superposition of coherent states. In particular, for an initial superposition of two coherent states of the same amplitude,

$$\mathcal{N}^{-1}\left(|\alpha e^{i\psi}\rangle + |\alpha e^{-i\psi}\rangle\right) \tag{9.11}$$

where $\mathcal{N}^{-1}$ is the normalization constant, we obtain from Equation 9.8 that the nondiagonal elements of the density matrix in the resonant case, $\delta = 0$, rapidly decrease as

$$|(\alpha e^{i\psi} + b^*)e^{-\gamma t/2}\rangle\langle(\alpha e^{-i\psi} + b^*)e^{-\gamma t/2}|\exp\left[-|\alpha|^2(1-e^{-\gamma t})(1-e^{2i\psi})\right]$$

This means that for times $t \sim \gamma^{-1}|\alpha|^{-2}$ the coherence between the states (9.11) is lost and the field evolves into an incoherent superposition.

9.2
Dicke Model with Dissipation and Pumping (Dispersive Limit)

Now let us consider a collection of A two-level atoms interacting with a mode of the quantum field (Dicke model) inside a lossy cavity. Also, we suppose that the field is being pumped by an external (classical) field. The master equation that describes this situation is of the form (9.1) and (9.3), where the Hamiltonian of the system H is now

$$H = \omega_a S_z + \omega_f a^\dagger a + g\left(aS_+ + a^\dagger S_-\right) + \Omega\left(ae^{-i\omega_c t} + a^\dagger e^{i\omega_c t}\right) \tag{9.12}$$

and $S_{\pm,z}$ are the collective operators that form the $A+1$ dimensional representation of the $su(2)$ algebra. In the interaction frame, the Hamiltonian (9.12) takes the form

$$H = \Delta S_z + g\left(aS_+ + a^\dagger S_-\right) + \Omega\left(ae^{it\delta} + a^\dagger e^{-i\delta t}\right) \tag{9.13}$$

where $\delta = \omega_f - \omega_c$ and $\Delta = \omega_a - \omega_f$. We consider the dispersive interaction limit between the field and the atomic system, when $A\sqrt{\overline{n}}g \ll \Delta$ is satisfied, where $\overline{n}$ is the average number of photons in the field. In this limit, as we already know, we can eliminate the transitions between different eigenstates of the operator S_z adiabatically by applying the transformation (7.94) to the Hamiltonian (9.12).

Retaining terms of the order $g/\Delta \ll 1$, we obtain (in the frame rotating with the classical field frequency ω_c)

$$\begin{aligned} H_1 = {} & \Delta_1 S_z + \delta a^\dagger a + \frac{g^2}{\Delta}\left[\left(2a^\dagger a + 1\right) S_z + S^2 - S_z^2\right] \\ & + \Omega\left[a + a^\dagger + \frac{2\sqrt{2}g}{\Delta} S_x\right] \end{aligned} \tag{9.14}$$

where $\Delta_1 = \omega_a - \omega_c$ and $S^2 = (A/2 + 1)\,A/2$ is the Casimir operator of the $su(2)$ algebra. It can easily be seen that the term $\sqrt{8}\Omega g S_x/\Delta$ in the above equation is eliminated by applying the following transformation to the Hamiltonian H_1:

$$V = \exp\left(-\mathrm{i}\frac{2\sqrt{2}\Omega g}{\Delta_1^2} S_y\right)$$

This does not affect the rest of the Hamiltonian (within the given accuracy and when the amplitude of the external field is not very high, $\Omega g \ll \Delta_1^2$). Finally, the effective Hamiltonian takes the form

$$H_{\text{eff}} = \Delta_1 S_z + \delta a^\dagger a + \eta\left(2a^\dagger a + 1\right) S_z + \eta\left(S^2 - S_z^2\right) + \Omega\left(a + a^\dagger\right) \tag{9.15}$$

where $\eta = g^2/\Delta$. The transformation V (like U, see Section 7.8) is time independent and does not affect the dynamics of the observables. These transformations do not affect the dissipative part (Equation 7.39) of the master equation (Equation 9.1) either, in the case of weak dissipation, $\gamma \ll g$, since they would add terms of order $O(\gamma g/\Delta)$ to the equation. Thus, the master equation that describes the Dicke model with dissipation and pumping field in the dispersive limit has the form (9.1), where the Hamiltonian is given by Equation 9.15. This Hamiltonian can be simplified still further. The Hamiltonian that describes the evolution of the density matrix transformed according to

$$\widetilde{\rho} = \mathrm{e}^{\mathrm{i}(\Delta_1 S_z + \eta(S^2 - S_z^2))t} \rho \mathrm{e}^{-\mathrm{i}(\Delta_1 S_z + \eta(S^2 - S_z^2))t}$$

is of the form

$$\widetilde{H}_{\text{eff}} = \delta a^\dagger a + \eta\left(2a^\dagger a + 1\right) S_z + \Omega\left(a + a^\dagger\right) \tag{9.16}$$

The Hamiltonian (9.16) is diagonal in the atomic space, $[\widetilde{H}_{\text{eff}}, S_z] = 0$, and linear in the field space. Thus, the master equation can be solved by using the same method as in the previous section. In the bare atomic basis

$$S_z|k\rangle_{\text{at}} = \left(k - \frac{A}{2}\right)|k\rangle_{\text{at}}$$

the master equation takes the linear form on the operators (9.5)

$$\partial_t \widetilde{\rho}_{pq} = \left[\gamma N_- + v_p N_l + v_q^* N_r + \mathrm{i}\Omega\left(A_r + A_{+r} - A_l - A_{+l}\right)\right]\widetilde{\rho}_{pq} \tag{9.17}$$

where $\widetilde{\rho}_{pq} = {}_{\mathrm{at}}\langle p|\tilde{\rho}|k\rangle_{\mathrm{at}}$ and

$$\upsilon_p = -\mathrm{i}\delta - 2\mathrm{i}\eta\lambda_p - \frac{\gamma}{2}, \; \lambda_p = p - \frac{A}{2}$$

Recall that the matrix elements $\widetilde{\rho}_{pq}$ are still operators in the field space. The solution of Equation 9.17 is given by

$$\widetilde{\rho}_{pq}(t) = \mathrm{e}^{\mu_{pq}}\mathrm{e}^{\upsilon_q^* t N_r}\mathrm{e}^{\upsilon_p t N_l}\mathrm{e}^{f_{pq}N_-}\mathrm{e}^{b_p^* A_{+l}}\mathrm{e}^{c_{pq}A_l}\mathrm{e}^{b_q A_r}\mathrm{e}^{c_{qp}^* A_{+r}}\widetilde{\rho}_{pq}(0) \tag{9.18}$$

where $\widetilde{\rho}_{pq}(0)$ is the initial matrix and the parameters are sought in a form analogous to Equation 9.9, which yields

$$\begin{aligned}\frac{\mu_{pq}(t)}{\Omega^2} &= \chi_{pq}\, t + \frac{\gamma}{(\upsilon_p + \upsilon_q^*)\upsilon_p\upsilon_q^*}\left(\mathrm{e}^{-\upsilon_p t} - 1\right)\left(\mathrm{e}^{-\upsilon_q^* t} - 1\right) \\ &\quad - \left(1 + \frac{\gamma}{\upsilon_p + \upsilon_q^*}\right)\left(\frac{\mathrm{e}^{\upsilon_p t} - 1}{\upsilon_p^2} + \frac{\mathrm{e}^{\upsilon_q^* t} - 1}{\upsilon_q^{*2}}\right) \\ \chi_{pq} &= \frac{2\eta}{|\upsilon_p|^2|\upsilon_q|^2}\left[\mathrm{i}(\lambda_q - \lambda_p)\left(\frac{\gamma^2}{4} + (\delta + 2\eta\lambda_p)(\delta + 2\eta\lambda_q)\right)\right. \\ &\quad \left. - \gamma\eta(\lambda_p - \lambda_q)^2\right]\end{aligned} \tag{9.19}$$

$$\begin{aligned} f_{pq}(t) &= \frac{\gamma}{\upsilon_p + \upsilon_q^*}\left[\mathrm{e}^{(\upsilon_p + \upsilon_q^*)t} - 1\right], \; b_q(t) = -\mathrm{i}\frac{\Omega}{\upsilon_q^*}(\mathrm{e}^{-\upsilon_q^* t} - 1), \\ c_{pq}(t) &= -\mathrm{i}\Omega\left[\left(1 + \frac{\gamma}{\upsilon_p + \upsilon_q^*}\right)\frac{\mathrm{e}^{\upsilon_p t} - 1}{\upsilon_p} + \frac{\gamma}{(\upsilon_p + \upsilon_q^*)\upsilon_q^*}(\mathrm{e}^{-\upsilon_q^* t} - 1)\right]\end{aligned} \tag{9.20}$$

We consider first the purely dissipative case, $\Omega = 0$. Starting from an arbitrary initial atomic state, and a coherent state of the field, from Equation 9.18 we obtain for the density matrix of the system

$$\widetilde{\rho}(t) = \sum_{p,q=0}^{A} c_p c_q^* |p\rangle_{\mathrm{at\,at}}\langle q| \otimes |\alpha\,\mathrm{e}^{\upsilon_p t}\rangle_{\mathrm{f\,f}}\langle\alpha\,\mathrm{e}^{\upsilon_q t}|\exp\left[|\alpha|^2\left(f_{pq} + \mathrm{e}^{-\gamma t} - 1\right)\right] \tag{9.21}$$

where f_{pq} is defined in Equation 9.20. We may observe that for long times ($t \gg \gamma^{-1}$), the atomic density matrix takes the form

$$\widetilde{\rho}_{\mathrm{at}}(t) = \sum_{p,q=0}^{A} c_p c_q^* \mid p\rangle_{\mathrm{at\,at}}\langle q| \, \exp\left[-2\mathrm{i}\eta|\alpha|^2\frac{\lambda_p - \lambda_q}{\gamma + 2\mathrm{i}\eta\left(\lambda_p - \lambda_q\right)}\right] \tag{9.22}$$

while the field evolves to the vacuum state: $\rho_{\mathrm{f}} \to |0\rangle_{\mathrm{f\,f}}\langle 0|$. This means that the atomic coherence is lost only partially in the presence of the dissipation of the field and the degree of decoherence decreases when the parameter γ increases. Considering the evolution of the nondiagonal elements of the density matrix to their stationary state, we note that the factor $\mathrm{e}^{-\gamma t}$ determines the envelope of oscillations with frequency $2\eta(\lambda_p - \lambda_q)$, while the first oscillation has a Gaussian envelope $\exp\left[-2t^2\eta^2|\alpha|^2\left(\lambda_p - \lambda_q\right)^2\right]$, which does not depend on γ. For certain values of the parameters $\eta|\alpha|$ and γ, it is possible that only one oscillation survives. In such cases, we would have a Gaussian decay with a decay time that does not depend on γ (while the stationary state, of course, depends on γ).

In the presence of a pumping field, we find that, for the initial coherent state of the field $\rho_f(0) = |\alpha\rangle_{\mathrm{f}\,\mathrm{f}}\langle\alpha|$ and an arbitrary atomic state

$$|\mathrm{in}\rangle_{\mathrm{at}} = \sum_{k=0}^{A} c_p|p\rangle$$

the atomic density matrix for the characteristic times $t \gg \gamma^{-1}$ (when $\mathrm{e}^{-\gamma t} \ll 1$) acquires a similar form to the one in Equation 9.22:

$$\widetilde{\rho}_{\mathrm{at}}(t) = \sum_{p,q=0}^{A} c_p c_q^* |p\rangle_{\mathrm{at}\,\mathrm{at}}\langle q| \exp\left[\Omega^2\chi_{pq}t\right]\widetilde{p}_{\mathrm{at}}^{pq} \tag{9.23}$$

where

$$\widetilde{p}_{\mathrm{at}}^{pq} = \exp\left[\left(\Omega^2\frac{v_p^2 + v_q^{*2} + v_p v_q^*}{(v_p v_q^*)^2} - |\alpha|^2\right)\left(1 + \frac{\gamma}{v_p + v_q^*}\right)\right]$$
$$\times \exp\left[\mathrm{i}\Omega\left(\frac{\alpha}{v_p} - \frac{\alpha^*}{v_q^*}\right)\left(1 + \frac{\gamma}{v_p + v_q^*}\right)\right]$$

(the time-dependent factor is discussed later). It can easily be seen that $\widetilde{p}_{\mathrm{at}}^{pp} = 1$, which means that there is no population transfer, as is expected in the dispersive regime. The field density matrix takes the form of an incoherent superposition of the coherent states:

$$\rho_{\mathrm{f}}(t) = \sum_{p=0}^{A} |c_p|^2 |z_p(t)\mathrm{e}^{-\mathrm{i}\omega_c t}\rangle_{\mathrm{f}\,\mathrm{f}}\langle z_p(t)\mathrm{e}^{-\mathrm{i}\omega_c t}|$$

where

$$z_p(t) = \left(\alpha + b_p^*(t)\right)\mathrm{e}^{v_p t}$$

and $b_p^*(t)$ is defined as in Equation 9.20. In the stationary state ($t \to \infty$), the field density matrix acquires the form

$$\rho_{\mathrm{f}}^{ss} = \sum_{p=0}^{A} |c_p|^2 |z_p^{ss}\mathrm{e}^{-\mathrm{i}\omega_c t}\rangle_{\mathrm{f}\,\mathrm{f}}\langle z_p^{ss}\mathrm{e}^{-\mathrm{i}\omega_c t}|$$

with $z_p^{ss} = \mathrm{i}\Omega/v_p$. The average number of photons in the stationary state would thus be

$$\langle\hat{n}\rangle = \mathrm{Tr}\left(a^\dagger a\,\rho_{\mathrm{f}}^{ss}\right) = \Omega^2\sum_{p=0}^{A}\frac{|c_p|^2}{\left(\delta + 2\eta\lambda_p\right)^2 + \gamma^2/4}$$

Note that the stationary state of the field depends on the atomic indices.

From the expression (9.23) we observe that the atomic density matrix for the typical dissipative decoherence times $t \sim \gamma^{-1}$ still depends on time through the factor

$$\exp\left[\Omega^2 \chi_{pq} t\right]$$

which equals unity for $p = q$ and decreases when $p \neq q$. This factor leads to total loss of coherence between the atomic components (without mixing the atomic populations) for the times

$$t_{\text{dec}}^{pq} \sim \frac{|v_p|^2 |v_q|^2}{\gamma \Omega^2 \eta^2 \left(\lambda_p - \lambda_q\right)^2}, \quad p \neq q$$

For typical values of the parameters of the system, $\eta \ll \gamma$ in resonance $\delta = 0$, the decoherence time is of order

$$t_{\text{dec}} \sim \frac{\gamma^3}{\Omega^2 \eta^2} \tag{9.24}$$

which is usually much larger than the dissipative decoherence time. This implies that for the times $t \sim t_{\text{dec}}$ the atomic system will be found in a completely mixed state:

$$\rho_{\text{at}}(t \succeq t_{\text{dec}}) \simeq \sum_{p=0}^{A} |c_p|^2 |p\rangle_{\text{at}\,\text{at}}\langle p|$$

Thus, all the collective effects that arise from the nonlinearity of the effective Hamiltonian (9.15), disappear for $t \succeq t_{\text{dec}}$.

Note that pure dissipation does not cause total loss of atomic coherence. It is precisely the simultaneous action of the dissipation and the pumping field that leads to a complete decay of the nondiagonal elements of the atomic density matrix.

Thus, there are two timescales in which the atomic coherence is lost: first, the coherence is partially lost for times of the order of γ^{-1} owing to the field dissipation; then, for times t_{dec} (Equation 9.24), the atomic matrix acquires a diagonal form with the initial distribution of the atomic population. It is important to note that the neglected terms in the deduction of the effective Hamiltonian (9.15) also produce a loss of coherence, which, apart from the decay of nondiagonal elements, leads to the randomization of the population distribution.

9.3
Dicke Model with Dissipation (Resonant Case)

In this section, we analyze the resonant Dicke model in the presence of the field dissipation mechanism (9.3) [105, 106] (see also [107]) at zero temperature. The equation for the density matrix that describes the evolution of the system has the form

$$\partial_t \rho = -\mathrm{i}g[aS_+ + a^\dagger S_-, \rho] + \frac{\gamma}{2}(2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a) \tag{9.25}$$

where γ is the decay rate. During the dissipative evolution, the field loses photons, so that the total number of excitations is no longer conserved. However, if the system is initially prepared in a highly excited field state, then, for not very long times the system continues in a highly excited state. In this period the field distribution does not reach the vacuum state; in order to solve Equation 9.25 we can apply the technique used in Chapter 7. Applying the Q-transformation (7.1) to the Equation 9.25 we obtain in the zeroth order approximation (7.8) for the transformed density matrix

$$\underline{\rho} = \hat{Q}^{\dagger}\rho\hat{Q} \tag{9.26}$$

the following expression:

$$\partial_t\underline{\rho} = -2\mathrm{i}g[\sqrt{\hat{\nu}+1/2}\,S_x, \underline{\rho}] + L_0\underline{\rho} + L_1\underline{\rho} \tag{9.27}$$

with

$$L_0\underline{\rho} = \frac{\gamma}{2}\left(2a\underline{\rho}a^{\dagger} - a^{\dagger}a\underline{\rho} - \underline{\rho}a^{\dagger}a\right)$$

$$L_1\underline{\rho} = \frac{\gamma}{2}\left(2\underline{a}\underline{\rho}\underline{a}^{\dagger} - 2a\underline{\rho}a^{\dagger} + \left\{S_z + \frac{A}{2}, \underline{\rho}\right\}\right)$$

The transformed Equation 9.27 is diagonal in the field space and is very convenient for numerical calculations. To find an analytical solution for Equation 9.27, we observe that the operator L_1 can be neglected if the average number of field photons is very large, since

$$L_1\underline{\rho} = \frac{\gamma}{2}\left(-\left\{\frac{S_z + A/2}{(\hat{n}+1)}, a\underline{\rho}a^{\dagger}\right\} + \left\{S_z + \frac{A}{2}, \underline{\rho}\right\}\right) + O\left(\frac{\gamma}{\hat{n}+1}\right)$$

so that $L_1\underline{\rho} \approx O\left(\gamma/\hat{n}\right)$, where the condition $\gamma \ll g\bar{n}$ should also be fulfilled in this regime. In the number state basis, Equation 9.27 has the form

$$\partial_t\underline{\rho}_{nm} = -2\mathrm{i}g(S_x\sqrt{n_A}\,\underline{\rho}_{nm} - \underline{\rho}_{nm}\sqrt{m_A}\,S_x) + (L_0\underline{\rho})_{nm} \tag{9.28}$$

where $n_A = n + 1/2 - A/2$. The matrix elements $\underline{\rho}_{nm}$ continue to be operators in the atomic space. Equation 9.28 takes a simple form in the semiclassical atomic basis (where the operator S_x is diagonal):

$$S_x|\underline{p}\rangle_{\mathrm{at}} = \lambda_p|\underline{p}\rangle_{\mathrm{at}}, \quad \lambda_p = p - \frac{A}{2}, \quad p = 0, 1, \ldots, A \tag{9.29}$$

The representation of bare atomic states in the semiclassical atomic basis $|\underline{p}\rangle_{\mathrm{a}}$ is:

$$|k\rangle_{\mathrm{at}} = \sum_p C_{kp}|\underline{p}\rangle_{\mathrm{at}} \tag{9.30}$$

where the matrix elements C_{kp}

$$C_{kp} = \langle\underline{p}|k\rangle = \langle p|\exp\left(\frac{\mathrm{i}\pi S_y}{2}\right)|k\rangle = \mathrm{i}^{k-p}P^{A}_{pk}\left(\frac{\pi}{2}\right) \tag{9.31}$$

where $P^A_{pk}(\theta) = \langle p| \exp(\mathrm{i}\theta S_x)|k\rangle$ (see Appendix 11.5) and the inverse transformation has the form $|\underline{p}\rangle_{ai} = \sum_p C^*_{pk}|k\rangle_{ai}$, $C^*_{pk} = \langle k|\underline{p}\rangle = C_{kp}$.

For the matrix elements of $\underline{\rho}_{nm}$ in the semiclassical basis, $x_{qp,nm} = \langle \underline{q}|\underline{\rho}_{nm}|\underline{p}\rangle$ Equation 9.28 takes the form

$$\partial_t x_{qp,nm} = -f^{qp}_{nm} x_{qp,nm} + \gamma\sqrt{(n+1)(m+1)}\, x_{qp,n+1m+1} \tag{9.32}$$

where

$$f^{qp}_{nm} = 2\mathrm{i}g\left(\lambda_q\sqrt{n_A} - \lambda_p\sqrt{m_A}\right) + \gamma\frac{(n+m)}{2} \tag{9.33}$$

Making the following change of variable (omitting the atomic indexes in this calculation)

$$x_{nm} = \frac{y_{nm}}{\gamma^n\sqrt{n!m!}}$$

we obtain

$$\partial_t y_{nm} = -f_{nm} y_{nm} + y_{n+1m+1}$$

and applying the Laplace transform we arrive at the following recurrence equation:

$$\bar{y}_{n+1m+1}(s) - (f_{nm} + s)\bar{y}_{nm}(s) = -\bar{y}_{nm}(0)$$

where $\bar{y}_{nm}(s)$ is the Laplace transform of y_{nm}. The solution of the above equation is

$$\bar{y}_{nm}(s) = \sum_{k=0}^{\infty} \frac{\bar{y}_{n+k\,m+k}(0)}{\prod_{j=0}^{k}(f_{n+j\,m+j} + s)} \tag{9.34}$$

whose inverse transform is

$$y_{nm}(t) = \sum_{k=0}^{\infty} y_{n+k\,m+k}(0) I^{(q,p)}_{k,\,nm}(t) \tag{9.35}$$

The solution of Equation 9.32 then takes the form

$$x_{qp,nm}(t) = \sum_{k=0}^{\infty} \gamma^k \sqrt{\frac{(n+k)!}{n!}\frac{(m+k)!}{m!}}\, x_{qp,n+k\,m+k}(0)\, I^{(qp)}_{k,nm}(t) \tag{9.36}$$

where

$$I^{(qp)}_{0,nm}(t) = \exp(-f_{qp,nm}t)$$

$$I^{(qp)}_{k,nm}(t) = \sum_{j=0}^{k} \frac{\exp(-f^{qp}_{n+j\,m+j}t)}{\prod_{\mathrm{i}=0}^{k}(f^{qp}_{n+\mathrm{i}\,m+\mathrm{i}} - f^{qp}_{n+j\,m+j})|_{\mathrm{i}\neq j}}, \quad k > 0 \tag{9.37}$$

The initial condition $x_{qp,nm}(0)$ is convenient to express in terms of bare initial conditions:

$$x_{qp,nm}(0) = {}_a\langle \underline{q}|\langle n|\hat{Q}^\dagger \rho(0)\hat{Q}\,|m\rangle|\underline{p}\rangle_a = \sum_{k,l} C_{kq} C_{lp} \rho_{kl,n-k\,m-l}(0) \tag{9.38}$$

where $\rho(0)$ is the density matrix, describing any initial pure or mixed state of the whole system. The expressions (9.36) to (9.38) provide an exact solution for the Equation 9.32.

The analytic form (9.36) allows to describe the influence of the dissipation for the initial condition of a strong field. The dynamics of the Dicke model in a cavity without losses are obtained by selecting $\gamma = 0$. In this case, only the term $k = 0$ contributes to the sum (9.36). We can also obtain the evolution of the quantum field in a cavity with dissipation by letting $g = 0$ in the solution (9.36).

9.3.1
Initial Field Number State

Now we suppose that the field is initially in the number state $|N\rangle_{\text{f}}$ (with $N \gg A$) and that the atomic system is in a nonexcited state, the density matrix of the total system then has the form

$$\rho(0) = |0\rangle_{\text{at at}}\langle 0| \otimes |N\rangle_{\text{f f}}\langle N|$$

and we obtain from Equation 9.38

$$x_{qp,nm}(0) = \langle \underline{p}|0\rangle \langle \underline{q}|0\rangle \delta_{nN}\, \delta_{mN} \tag{9.39}$$

Substituting Equation 9.39 in Equation 9.36 we obtain the matrix elements of the Q-transformed density matrix in the semiclassical basis:

$$x_{qp,nm}(t) = \gamma^{N-n} \frac{N!}{n!} I^{(qp)}_{N-n,nn}(t)\; C_{0q} C_{0p}\, \delta_{mn} \tag{9.40}$$

and in the bare atomic basis:

$$\underline{\rho}_{kl,nm}(t) = \delta_{mn} \sum_{q,p=0}^{A} C_{kq}\, C_{lp}\, x_{qp,nn}(t) \tag{9.41}$$

In this case, we observe that the density matrix is diagonal in the field space.

The atomic inversion can easily be evaluated by using Equations 9.40 and 9.41,

$$\langle S_z \rangle = -\frac{A}{2} \text{Re} \sum_n \gamma^{N-n} \frac{N!}{n!} I^{(p\,p-1)}_{N-n,nn}$$

where $I^{(p\,p-1)}_{N-n,nn}$ is defined in (9.37) and is independent of the atomic index p.

9.3.2
Initial Field Coherent State

If the field is initially in a coherent state $\overline{n} \gg A$, then the functions (9.33) vary slowly near $\overline{n}$ owing to the large number of photons, and the following approximation can be made:

$$f^{qp}_{n+j,m+j} \simeq f^{qp}_{nm} + j\gamma'_{qp,nm}$$

where

$$\gamma'_{qp,nm} = \gamma + \mathrm{i}g\left(\frac{\lambda_q}{\sqrt{n_A}} - \frac{\lambda_p}{\sqrt{m_A}}\right) \tag{9.42}$$

Using Equation 9.42, the coefficients in Equation 9.37 are simplified, and one obtains

$$I^{(qp)}_{k,nm}(t) = \frac{1}{k!}\left[\frac{1 - \exp(-\gamma'_{qp,nm}t)}{\gamma'_{qp,nm}}\right]^k \exp(-f^{qp}_{qp,nm}t) \tag{9.43}$$

For the initial field coherent state $|\alpha\rangle$ with zero phase and the atomic system in an arbitrary state

$$|\mathrm{in}\rangle_a = \sum_p c_p|\underline{p}\rangle_a$$

so that, the initial density matrix is $\rho(0) = |\mathrm{in}\rangle\langle\mathrm{in}| \otimes |\alpha\rangle\langle\alpha|$, we obtain from Equation 9.38

$$x_{qp,nm}(0) = \sum_{ll'=0}^{A}\sum_{rr'=0}^{A} C_{lq}C_{l'p}C_{lt}C_{l'r'}c_r c^*_{r'}\, p_{n-l}p^*_{m-l'} \tag{9.44}$$

where C_{lp} is defined by Equation 9.31 and p_n is the square root of the Poisson distribution,

$$p_n = \mathrm{e}^{-\overline{n}/2}\,\frac{\overline{n}^{n/2}}{\sqrt{n!}}$$

The form of the initial condition (9.44) can be simplified by using the following approximation:

$$p_{n+l} = p_n + O\left(\frac{l}{\sqrt{\overline{n}}}\right) \tag{9.45}$$

where $l \sim A$, so that neglecting the terms of order $O(A/\sqrt{\overline{n}})$ and considering that

$$\sum_l C_{lr}C_{lq} = \delta_{rq}$$

we obtain

$$x_{qp,nm}(0) \approx c_q\, c^*_p\, p_n p_m \tag{9.46}$$

The above equation is exact when the initial state of the atomic system is nonexcited, so that $c_p = C_{0p}$, and the transformation $\hat{Q}$ is reduced to the identity operator.

Substituting the initial condition (9.46) and the coefficients (Equation 9.43) into Equation 9.36, we obtain a simple expression for the matrix elements of the density operator in the quasi-classical basis:

$$x_{qp,nm}(t) = c_q\, c^*_p\, p_n(t)p_m(t)F_{qp,nm}(t) \tag{9.47}$$

where $p_n(t)$ is the square root of the Poisson distribution with the average of the number of photons depending on time:

$$p_n(t) = \mathrm{e}^{-\overline{n}(t)/2} \frac{\overline{n}(t)^{n/2}}{\sqrt{n!}} \tag{9.48}$$

$$\overline{n}(t) = \overline{n}\mathrm{e}^{-\gamma t} \tag{9.49}$$

and the function $F_{qp,nm}(t)$ is given by,

$$F_{qp,nm}(t) = \mathrm{e}^{-2\mathrm{i}gt(\lambda_q\sqrt{n_A}-\lambda_p\sqrt{m_A})-\mu_{qp}(t)} \tag{9.50}$$

$$\mu_{qp}(t) = \overline{n}\left[(1-\mathrm{e}^{-\gamma t}) - \frac{\gamma}{\gamma'}(1-\mathrm{e}^{-\gamma' t})\right] \tag{9.51}$$

$$\gamma' = \gamma + \frac{\mathrm{i}g(\lambda_q - \lambda_p)}{\sqrt{\overline{n}(t) - A/2 + 1/2}} \tag{9.52}$$

To obtain the above expression, the function $\gamma'_{qp,nm}$ (defined in Equation 9.42) is approximated by its value at the point $n = m = \overline{n}(t)$.

Finally, the Q-transformed density matrix (9.47) in the bare atomic basis acquires the form

$$\underline{\rho}_{kl,nm}(t) = p_n(t)p_m(t)\sum_{q,p=0}^{A} C_{kq}\, C_{lp}\, c_q\, c_p^*\, F_{pq,nm}(t) \tag{9.53}$$

Using Equation 9.26, we may observe that the density matrix in the bare atomic basis is connected to $\underline{\rho}_{kl,nm}$ by the following simple relation:

$$\rho_{kl,nm}(t) = \underline{\rho}_{kl,n+k,m+l}(t) \tag{9.54}$$

Equations 9.47 to 9.52 describe the dissipative dynamics in the case of the strong coherent field. Now, it is easy to calculate the expectation value of any observable. We find, for example, the atomic inversion when the atoms are initially prepared in the nonexcited state. The atomic density matrix in the bare atomic basis has the form

$$\rho_{kl}^{(at)} = \sum_n \underline{\rho}_{kl,k+n\,l+n}(t)$$

and, hence, we find the atomic inversion evolution (see Figure 9.1),

$$\langle S_z(t)\rangle = \sum_{k=0}^{A}\left(k - \frac{A}{2}\right)\rho_{kk}^{(at)} = \sum_n \sum_{k=0}^{A}\left(k - \frac{A}{2}\right)\underline{\rho}_{kk,k+n\,k+n}(t)$$

where $\underline{\rho}_{kk,k+n\,k+n}(t)$ is defined in Equation 9.53 with

$$c_p = C_{0p} = \mathrm{i}^{-p} P_{0p}^{A}\left(\frac{\pi}{2}\right) = \sqrt{\frac{A!}{(A-k)!k!}}\sin^k\frac{\theta}{2}\cos^{A-k}\frac{\theta}{2}$$

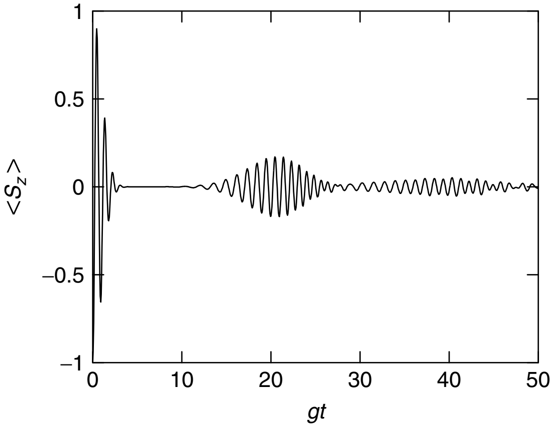

Figure 9.1 Evolution of the atomic inversion in the resonant Dicke model in the presence of dissipation: decreasing of the amplitude of the first revival; the field is initially in a coherent state with $\overline{n} = 12$ and the atoms, $A = 2$, are initially excited; $\gamma/g = 0.005$.

9.3.3
Factorized Dynamics

As we know, the evolution of the semiclassical atomic states in the strong coherent field leads to the wave function factorization phenomenon. We now prove that in the dissipative case an analog of such factorization is preserved. We consider the evolution of the initial atomic matrix $\rho^{(\mathrm{at})}(0) = |\underline{q}\rangle\langle\underline{p}|$ in the coherent field: $\rho^{[qp]}(0) = |\underline{q}\rangle\langle\underline{p}| \otimes |\alpha\rangle\langle\alpha|$. From Equations 9.53 and 9.54 we obtain the following expression for the density matrix in the bare atomic basis:

$$\rho^{[qp]}_{kl,nm}(t) = p_{n+k}(t)p_{m+l}(t)C_{kq}\,C_{lp}\,F_{pq,n+k\,m+l}(t) \tag{9.55}$$

For not very long evolution times, while $A \ll \sqrt{\overline{n}}\mathrm{e}^{-\gamma t}$, the time-dependent Poisson distribution in Equation 9.55 can be approximated as (9.45):

$$p_{n+k}(t) \approx p_n(t) + O\left(\frac{A}{\sqrt{\overline{n}(t)}}\right) \tag{9.56}$$

Then, we expand the square roots of the frequencies appearing in the functions $F_{qp,n+k\,m+k}$ as follows:

$$\sqrt{n_A + k} \approx \sqrt{n_A} + \frac{k}{2\sqrt{n_A}} + O\left(\frac{A^2}{\overline{n}^{3/2}}\right) \tag{9.57}$$

Substituting Equations 9.56 and 9.57 into Equation 9.55 we obtain

$$\rho^{[qp]}_{kl,nm} \approx \exp(-\mu_{qp})p_n(t)p_m(t)\,C_{kq}C_{lp} \\ \times \exp\left(-2\mathrm{i}gt(\lambda_q\sqrt{n_A} - \lambda_p\sqrt{m_A})\right)\exp\left(-\mathrm{i}gt\left(\frac{k\lambda_q}{\sqrt{n_A}} - \frac{l\lambda_p}{\sqrt{m_A}}\right)\right) \tag{9.58}$$

Now, let us make a further approximation (the same as we did in the nondissipative case): we approximate the slow frequencies in the exponentials in Equation 9.58 as

$$\frac{k}{\sqrt{n_A}} \approx \frac{k}{\sqrt{\overline{n}_A(t)}} + O\left(\frac{A}{\overline{n}(t)}\right) \tag{9.59}$$

With this approximation, the accuracy is reduced even more, but we obtain a very clear physical representation: the total density matrix acquires the form of a product of the atomic density matrices and the field, each of which represents pure states of the atomic system and of the field:

$$\rho^{[qp]}(t) \approx \exp(-\mu_{qp}(t))|\Phi_q(t)\rangle\langle\Phi_p(t)| \otimes |A_q(t)\rangle\langle A_p(t)| \tag{9.60}$$

where the atomic and the field states appearing in the factorized density matrix are given as

$$\begin{aligned} |\Phi_q(t)\rangle &= \sum_n p_n(t) \exp\left[-2\mathrm{i}gt\lambda_q\sqrt{n_A}\right]|n\rangle_f \\ |A_q(t)\rangle &= \exp\left[-\mathrm{i}\,\frac{gt\lambda_q}{\sqrt{\overline{n}_A(t)}}\,(S_z + A/2)\right]|\underline{q}\rangle \end{aligned} \tag{9.61}$$

This solution differs from the one that we obtained in the nondissipative case in two points: *a*) the Poisson distribution depends on the time-dependent average number of photons $\overline{n}(t) = \overline{n}e^{-\gamma t}$; *b*) density matrices that correspond to different initial conditions $|\underline{p}\rangle\langle\underline{q}|$ are multiplied by factors $\exp(-\mu_{qp}(t))$. This does not affect the initial diagonal conditions in the semiclassical basis, since $\mu_{qq} = 0$. However, the factors $\exp(-\mu_{qp}(t))$ decrease rapidly for $q \neq p$, which results in a rapid loss of coherence between different semiclassical trajectories. From the explicit expression (Equation 9.51) for $\mu_{qp}(t)$, we can determine the decoherence time, that is, the time when different semiclassical states lose coherence between each other. This time is determined from the condition $\mu_{qp}(t_{\text{decoh}}) \sim 1$. For sufficiently strong initial fields, $\gamma\overline{n}^{3/2} \gg 1$, the decoherence time does not depend on the (initial) average number of photons $\overline{n}$ and is much smaller than the time of the first revival:

$$gt_{\text{decoh}}^{(qp)} \sim \left(\frac{6g}{\gamma\,(q-p)^2}\right)^{1/3} \ll gT_{\text{R}} \tag{9.62}$$

One can see that different semiclassical states lose mutual coherence more rapidly when $|q-p|$ increases. On the other hand, we note that in the case of weak dissipation, $\gamma \ll g$, we have $gt_{\text{decoh}}^{(qp)} \gg 1$, which means that the decoherence time is larger than the time of the collapse.

For an arbitrary initial atomic state, $|\text{in}\rangle = \sum_p c_q|\underline{q}\rangle$, the initial density matrix is represented as

$$\rho(0) = \sum_{qp} c_q c_p^* \rho^{[qp]}(0)$$

so that, the solution can immediately be obtained of the factorized form (Equations 9.60 and 9.61):

$$\rho(t) = \sum_{qp} c_q c_p^* \rho^{[qp]}(t) \tag{9.63}$$

For times longer than the decoherence time $t > t_{\text{decoh}}$, the field density matrix takes the form of an incoherent superposition

$$\rho_f(t) \approx \sum_p |c_p|^2 \rho_p(t), \quad \rho_p(t) = |\Phi_p(t)\rangle\langle\Phi_p(t)| \tag{9.64}$$

The factorization (9.60) allows us to understand the dynamics of the system in the phase space of the field: each semiclassical state is represented by a hump that moves along a spiral whose radius r decreases slowly with time according to $r(t) = \sqrt{\overline{n}}e^{-\gamma t/2}$. In the nondissipative case, two close semiclassical states coincide and produce the revival effect of the Rabi oscillations of the atomic inversion (see Section 7.3). In the dissipative case, these states rapidly lose their mutual coherence, which results in the suppression of the revival amplitude.

On the other hand, from Equations 9.62 and 9.63 we can reach the conclusion that the squeezing of the field, which takes place in the Dicke model, would not be affected either for long times or for short times. In effect, the squeezing of the field for short times is produced due to the interference between different semiclassical states and reaches its maximum value for times of the order of the collapse time $gt_{\text{sq}} \sim 1$, which is smaller than the decoherence time (Equation 9.62). On the other hand, the squeezing of the field for long times is due to the deformation of the field states associated with each of the atomic semiclassical states and is not related with the coherence between them. Thus, the squeezing can be destroyed only if the form of each ρ_p in Equation 9.64 is essentially changed. This could happen in the case of a strong dissipation $\gamma\sqrt{\overline{n}} \geq g$, when the average number of photons in the field decreases significantly for the time of the first revival: $\overline{n}(T_{\text{R}}) \ll \overline{n}$.

We can easily estimate the accuracy of the factorization (9.61). To this end, we evaluate the purity parameter $P = \text{Tr}\left(\rho_{\text{at}}\right)^2$ for an initial semiclassical atomic state $|\underline{q}\rangle\langle\underline{q}|$ using Equation 9.58:

$$P \approx \sum_{k,l=0}^{A} C_{qk}^2 C_{ql}^2 \exp\left[-4\overline{n}(t)\sin^2\frac{gt\,(k-l)}{8\overline{n}(t)^{3/2}}\right]$$

where C_{kp} are defined in Equation 9.31. It can easily be seen that the purity parameter essentially differs from unity (in the pure state $P = 1$) for the times of order $gt \sim \overline{n}(t)$. This means that the factorization (9.61) is valid for $gt \ll \overline{n}(t)$. In the case of weak dissipation, $\gamma\sqrt{\overline{n}} \ll g$, this condition leads to $gt \ll \overline{n}$. However, gt can be larger than $\sqrt{\overline{n}}$, which implies that the factorization survives for the times of various revivals.

9.4
Strong Dissipation

We consider an arbitrary quantum system X coupled to a cavity mode in a finite Q cavity. The master equation for the X-system-cavity mode density matrix is

$$\frac{d\rho}{dt} = -i\,[H, \rho] + L_a\rho + L_s\rho \tag{9.65}$$

The Hamiltonian $H = H_0 + V$ contains free field H_0, X-system evolution, and system–cavity mode interaction V, which are given by

$$H_0 = \omega_f a^\dagger a + \omega_0 X_0 \tag{9.66}$$

$$V = g\left(X_+ a + X_- a^\dagger\right) \tag{9.67}$$

where X_0 is the free system Hamiltonian and X_+, X_- are the operators describing transitions between X-system energy levels and, consequently, obeying the following commutation relations:

$$[X_0, X_\pm] = \pm X_\pm \tag{9.68}$$

We do not impose any condition on the commutator between transition operators X_+ and X_-, which is, in general, a function of the diagonal operator X_0, and, probably, some integrals of motion $\{N_i\}$ admitted by the X-system:

$$[X_+, X_-] = f(X_0, N_i) \tag{9.69}$$

The dissipation of the cavity mode coupling to the vacuum is described by the standard Lindblad-like term

$$L_a\rho = \frac{\gamma}{2}\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right) \tag{9.70}$$

where γ is the cavity damping rate. The term describing the X-system dissipation L_s can take on a different form (depending on the type of the X-system–reservoir coupling). Apart from the standard condition of the bad-cavity limit $g \ll \gamma$, we impose, for simplicity, the following restriction on the system parameters: $\chi \ll g^2/\gamma$, where χ is the X-system decay rate in L_s.

It is clear that, in the bad-cavity limit $g/\gamma \ll 1$, apart from the free Hamiltonian H_0, the dominant role in Equation 9.65 is due to the term (Equation 9.70) describing the dissipation of the field. Thus, to eliminate (in the first order of magnitude) the interaction term V from Equation 9.65, we apply the following (nonunitary) transformation to the equation:

$$U = \exp(iF) \tag{9.71}$$

where

$$F = \frac{g}{\gamma/2 - i\Delta}\left(L_1 - L_2 - M_1 + M_2\right) + \frac{g}{\gamma/2 + i\Delta}\left(L_{-1} - L_{-2} - M_1 + M_2\right)$$

and the operators $L_{\pm j}, M_{\pm j}, j = 1, 2, 3, 4$ are defined as

$$\begin{aligned} L_1\rho &= X_- a^\dagger \rho, & L_2\rho &= X_+ a\rho \\ L_{-1}\rho &= \rho X_- a^\dagger, & L_{-2}\rho &= \rho X_+ a \\ M_1\rho &= X_-\rho a^\dagger, & M_2\rho &= a\rho X_+ \end{aligned}$$

Making use of the following expansion,

$$U\rho = \rho + \mathrm{i}F\rho - \frac{1}{2!}F^2\rho + \cdots$$

and keeping terms up to the order $\left(g/\gamma\right)^2$, we obtain the equation for the transformed density matrix

$$\tilde{\rho} = U\rho$$

Here U is a super-operator acting on both sides, right and left, of the density matrix.

It follows from the form of the (super) operator F, that $\mathrm{Tr}\left(F^k\rho\right) = 0, k = 1, 2, \ldots,$ which indicates that the transformation (9.71) preserves the trace of the density matrix ($\mathrm{Tr}\tilde{\rho} = 1$).

In this approximation, the density matrix of the whole system satisfies the effective master equation:

$$\begin{aligned}\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} &= -\mathrm{i}\left[H_0, \tilde{\rho}\right] + L_a\tilde{\rho} \\ &+ \frac{g^2}{\gamma/2 - \mathrm{i}\Delta}\left(X_-\tilde{\rho}X_+ - X_+X_-\tilde{\rho} + af(X_0, N_i)\tilde{\rho}a^\dagger - a^\dagger af(X_0, N_i)\tilde{\rho}\right) \\ &+ \frac{g^2}{\gamma/2 + \mathrm{i}\Delta}\left(X_-\tilde{\rho}X_+ - \tilde{\rho}X_+X_- + a\tilde{\rho}f(X_0, N_i)a^\dagger - \tilde{\rho}a^\dagger af(X_0, N_i)\right)\end{aligned}$$

where $\tilde{\rho} = U\rho$, with U being the transformation defined by Equation 9.71 and $\Delta = \omega_0 - \omega_f$ is the detuning between the field and X-system frequencies.

Tracing over the field variables, we obtain the Lindblad-type [102] equation for the X-system reduced density matrix $\tilde{\rho}_X = \mathrm{Tr}_f\left(\tilde{\rho}\right)$

$$\frac{\mathrm{d}\tilde{\rho}_X}{\mathrm{d}t} = -\mathrm{i}\left[H_{\text{eff}}, \tilde{\rho}_X\right] + \alpha\frac{\gamma}{2}\left(2X_-\tilde{\rho}_X X_+ - X_+X_-\tilde{\rho}_X - \tilde{\rho}_X X_+X_-\right) \tag{9.72}$$

which describes the dissipation to the vacuum of the X system with the relaxation rate $\kappa = \alpha\gamma$, which is much smaller than the decay rate of the field γ. Here, the effective Hamiltonian has the form

$$H_{\text{eff}} = \omega_0 X_0 + \alpha\Delta\phi\left(X_0\right)$$

where $\phi\left(X_0\right) = X_+X_-$ and

$$\alpha = \frac{g^2}{(\gamma/2)^2 + \Delta^2}$$

As expected, the effective master equation for the X-system density matrix does not contain field operators. In principle, due to applied approximations, the master equation (Equation 9.72) holds for times $t \sim \gamma^2/g$ (when the X system is very close to its steady state).

It can be proved by direct substitution that the Laplace transform

$$\mathcal{L}\left(\tilde{\rho}_X\right)(s)=\int_0^{\infty} \mathrm{d}t \mathrm{e}^{-st} \tilde{\rho}_X(t)$$

of the solution of Equation 9.72 has the form [108]

$$\mathcal{L}\left(\tilde{\rho}_X\right)(s)=\sum_{k=0}^{\infty} \frac{(\alpha\gamma)^k}{\prod_{l=0}^k\left(s+\xi_l \otimes 1+1 \otimes \xi_l^{\dagger}\right)} X_-^k \tilde{\rho}_X(0) X_+^k \tag{9.73}$$

where $\xi_k \equiv \xi\left(X_0+k\right)$

$$\xi\left(X_0\right)=\alpha \frac{\gamma}{2} \phi\left(X_0\right)+\mathrm{i} H_{\mathrm{eff}}$$

where we have introduced the following notation:

$$(A \otimes B) \rho:=A \rho B$$

A and B being arbitrary operators.

The formal inverse Laplace transform of Equation 9.73 is

$$\begin{aligned}\tilde{\rho}_X(t)=&\sum_{l=0}^{\infty} \sum_{k=l}^{\infty}(\alpha\gamma)^k \frac{\mathrm{e}^{-\left(\xi_l \otimes 1+1 \otimes \xi_l^{\dagger}\right) t}}{\prod_{m=0,\, m \neq l}^k\left(\left(\xi_m-\xi_l\right) \otimes 1+1 \otimes\left(\xi_m-\xi_l\right)^{\dagger}\right)} \\ & \times X_-^k \tilde{\rho}_X(0) X_+^k\end{aligned} \tag{9.74}$$

9.4.1
Field–Field Interaction

Let us start by considering a single mode of the electromagnetic field as the X system. Then, we have

$$X_+=b^{\dagger}, \quad X_-=b, \quad X_0=b^{\dagger} b, \quad f\left(X_0\right)=-1$$

The Hamiltonian (9.66) takes the form

$$H=\omega_f a^{\dagger} a+\omega_0 b^{\dagger} b+g\left(a^{\dagger} b+b^{\dagger} a\right)$$

and could describe, for example, interaction between two coupled cavities, where one of them is strongly decaying to the vacuum.

We obtain the effective master equation for the mode "b" from Equation 9.72

$$\begin{aligned}\frac{\mathrm{d} \tilde{\rho}_b}{\mathrm{d}t}= & -\mathrm{i}\left(\omega_0+\alpha \Delta\right)\left[b^{\dagger} b, \tilde{\rho}_b\right] \\ & +\alpha \frac{\gamma}{2}\left(2 b \tilde{\rho}_b b^{\dagger}-b^{\dagger} b \tilde{\rho}_b-\tilde{\rho}_b b^{\dagger} b\right)\end{aligned} \tag{9.75}$$

It is worth noting here that the master equation (Equation 9.75), as well as in the general case (9.72), should be solved for the transformed initial conditions

$\tilde{\rho}_b(0) = U\rho_b(0)$, where U is defined in Equation 9.71. Nevertheless, this transformation represents just a small (nonunitary) rotation and, hence, it only generates small corrections of order g/γ to the initial state. Note that we did not impose any restriction on the relation g/ω_0 (which in both the microwave and optical region is, in general, very small).

9.4.2
Atom–Field Interaction

As another example, we consider a collection of A two-level atoms interacting with a single mode of the electromagnetic field inside a cavity (Dicke model); then

$$X_+ = S_+, \quad X_- = S_-, \quad X_0 = S_z, \quad f(X_0) = 2S_z$$

where the operators $S_\pm, S_z$ are generators of an $A+1$ dimensional representation of the $su(2)$ algebra, and the Hamiltonian (9.66) has the standard form

$$H = \omega_{\text{at}} S_z + \omega_f a^\dagger a + g\left(S_- a^\dagger + S_+ a\right)$$

From Equation 9.72 we immediately obtain the effective master equation for the atomic system:

$$\frac{\mathrm{d}\tilde{\rho}_{\text{at}}}{\mathrm{d}t} = -\mathrm{i}\left[H_{\text{eff}}, \tilde{\rho}_{\text{at}}\right] + \alpha\frac{\gamma}{2}\left(2S_-\tilde{\rho}_{\text{at}}S_+ - S_+S_-\tilde{\rho}_{\text{at}} - \tilde{\rho}_{\text{at}}S_+S_-\right) \tag{9.76}$$

The effective Hamiltonian is quadratic for the atomic population operator and has the following form:

$$H_{\text{eff}} = (\omega_{\text{at}} + \alpha\Delta)\,\omega_1 S_z - \alpha\Delta S_z^2 \tag{9.77}$$

9.5
Problems

9.1 Find the evolution of a coherent initial state of the field that is described by the following master equation (pure decoherence)

$$\dot{\varrho} = -\mathrm{i}\left[\omega a^\dagger a, \varrho\right] + \gamma a^\dagger a \varrho a^\dagger a - \frac{\gamma}{2}\left(a^\dagger a \varrho + \varrho a^\dagger a\right) \tag{9.78}$$

9.2 Find the operational solution of the master equation that describes the dissipation process for an atom with two energy levels,

$$\dot{\rho} = -\mathrm{i}\left[\omega\sigma_z, \rho\right] + \gamma\left(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-\right)$$

introducing the following operators

$$S_z = \frac{1}{2}\left(\sigma_z^r + \sigma_z^l\right), \quad S_+ = \sigma_+^r\sigma_-^l, \quad S_- = \sigma_-^r\sigma_+^l, \quad S_0 = \frac{1}{2}\left(\sigma_z^r - \sigma_z^l\right)$$

where $\sigma_{\pm,z}^r\rho = \sigma_{\pm,z}\rho$ and $\sigma_{\pm,z}^l\rho = \rho\sigma_{\pm,z}$.

Apply this solution to determine the decay of the excited atomic state $|1\rangle$ and of a coherent superposition $|\psi\rangle = (|1\rangle + |0\rangle)/\sqrt{2}$.

9.3 Prove that during the dissipative evolution process of the field in the Kerr medium, which is described by the following master equation

$$\dot{\varrho} = -\mathrm{i}\left[\omega(a^\dagger a)^2, \varrho\right] + \gamma a^\dagger a \varrho a^\dagger a - \frac{\gamma}{2}\left(a^\dagger a \varrho + \varrho a^\dagger a\right)$$

the average of the annihilation operator over the initial coherent state $|\alpha_0\rangle$ is

$$\langle a(t)\rangle = Tr(a\rho(t)) = \alpha_0 \exp\left(-gt + \mathrm{i}\,|\alpha_0|^2\,\omega\frac{(1-\mathrm{e}^{-2gt})}{g}\right)$$

where $g = \gamma/2 + \mathrm{i}\omega$.

10
Quasi-distributions in Quantum Optics

10.1
Quantization and Quasi-distributions

10.1.1
Weyl Quantization Method

A useful method for analyzing quantum optical systems consists of the mapping of system states into *c*-valued functions defined on the corresponding classical phase space. Such phase-space representations allow us to reformulate standard quantum mechanics in the classical language of phase spaces and functions defined on them [109], providing a very useful insight into quantum–classical correspondence in nonrelativistic quantum mechanics. According to this approach, both states and observables are considered to be functions on a given phase space, in such a way that average values are computed as in classical statistical mechanics, that is, by integrating some quasi-distribution function with a symbol of a corresponding operator over the phase space. The axiomatic approach to the phase space formulation of quantum mechanics was developed by Stratonovich [110]. Nowadays, this approach is known as *the Stratonovich–Moyal–Weyl correspondence*. According to this approach, we associate each operator $\hat{f}$ with its symbol $f(\xi)$, a *c*-number function defined on the corresponding phase space.

The map $\hat{f} \to f(\xi)$ is intimately related to the inverse map, $f(\xi) \to \hat{f}$, i.e. to the quantization problem: $\mathcal{M}$ being a differentiable manifold and $F^{\infty}(\mathcal{M})$ – a set of classical observables on $\mathcal{M}$ (real differentiable functions), we want to establish a linear correspondence:

$$F^{\infty}(\mathcal{M}) \to \mathrm{L}(\mathcal{H})$$

where $\mathcal{H}$ is a Hilbert space and $\mathrm{L}(\mathcal{H})$ is a set of Hermitian operators acting on $\mathcal{H}$

$$f(\xi) \to \hat{f}, \quad g(\xi) \to \hat{g}, \quad \hat{f}, \hat{g} \in \mathrm{L}(\mathcal{H})$$

which preserves the ‘‘commutation relations’’

$$\{f, g\} \to -\frac{i}{\hbar}[\hat{f}, \hat{g}], \quad \hbar \to 0 \tag{10.1}$$

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where $\{f, g\}$ means the Poisson brackets. It is well known that such correspondence cannot be found in a unique way. As one of the simplest and perhaps, best known examples, let us consider the case when the manifold $\mathcal{M}$ is a two-dimensional plane: $\mathcal{M} = \mathcal{R}^2$. In this case, it is easy to establish a "quantization" correspondence for linear and quadratic functions on the position and momentum variables:

$$q \to \hat{q}, \qquad p \to \hat{p}, \qquad [\hat{q}, \hat{p}] = i\hbar \tag{10.2}$$

$$q^2 \to \hat{q}^2, \qquad p^2 \to \hat{p}^2 \tag{10.3}$$

$$qp = \frac{1}{4}\left((q+p)^2 - (q-p)^2\right) \to \frac{1}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right) \tag{10.4}$$

Nevertheless, already for quartic polynomials of q and p such correspondence is not unique. Indeed, using Equation 10.3 we may obtain

$$q^2p^2 = \frac{1}{2}\left((q^2+p^2)^2 - q^4 - p^4\right) \to \frac{1}{2}\left(\hat{q}^2\hat{p}^2 + \hat{p}^2\hat{q}^2\right)$$

but, on the other hand, from Equation 10.4 we get

$$(qp)^2 \to \left(\frac{\hat{q}\hat{p} + \hat{p}\hat{q}}{2}\right)^2$$

Then, because

$$\frac{1}{2}\left(\hat{q}^2\hat{p}^2 + \hat{p}^2\hat{q}^2\right) \neq \left(\frac{\hat{q}\hat{p} + \hat{p}\hat{q}}{2}\right)^2$$

we cannot uniquely determine the operator that corresponds to q^2p^2. The absence of the unique correspondence also implies that the inverse mapping, from operators to c functions, is ill defined in such a simplified approach.

Several quantization methods have been proposed to avoid the above-mentioned difficulties. For our purpose, the idea of Weyl quantization (quantization with symmetries) is the most convenient. The main idea of this approach consists in the following: Let G be a transformation group of the manifold $\mathcal{M}$, so that G acts transitively on $\mathcal{M}$ ($\mathcal{M}$ is a homogenous space of G) and $\mathcal{H}$ is a Hilbert space where an irreducible unitary representation T_g of G is defined. We need to find for each f in $\mathcal{M}$ a correspondence

$$f(\xi) \to \hat{f}, \quad \hat{f} \in \mathrm{L}(\mathcal{H})$$

which preserves the commutation relation (10.1) and is covariant under the action of G, i.e.

$$f(g^{-1} \circ \xi) \to T_g \hat{f}\, T_g^{\dagger}, \quad g \in G$$

The last requirement is quite natural and allows us to establish an isomorphism between description of quantum systems either in terms of operators in Hilbert space or as c functions on the classical manifold $\mathcal{M}$.

To understand the main idea of Weyl quantization, we start with the simplest case of the two-dimensional plane: $\mathcal{M} = \mathcal{R}^2$, which is a homogeneous space for the Heisenberg–Weyl group $H(1)$. First, we establish a "quantization" correspondence for linear functions as in Equation 10.2. Next, we expand an arbitrary classical function of two canonical variables in the Fourier integral,

$$f(p,q) = \frac{1}{(2\pi)^2} \int \mathrm{d}x\,\mathrm{d}y\, \mathrm{e}^{-\mathrm{i}xq - \mathrm{i}yp} \int \mathrm{d}q'\mathrm{d}p'\, \mathrm{e}^{\mathrm{i}xq' + \mathrm{i}yp'} f(p', q')$$

The variables p and q enter linearly to the exponential, so that the quantization condition would be

$$\mathrm{e}^{-\mathrm{i}xq - \mathrm{i}yp} \to \hat{e}(x,y) = \mathrm{e}^{-\mathrm{i}x\hat{q} - \mathrm{i}y\hat{p}} \tag{10.5}$$

Then, the correspondence between a classical observable and the operator in the Hilbert space where an irreducible representation of the Heisenberg–Weyl group $H(1)$ is realized, is

$$f(p,q) \to \hat{f} = \frac{1}{(2\pi)^2} \int \mathrm{d}x\,\mathrm{d}y\, \hat{e}(x,y) \int \mathrm{d}q\,\mathrm{d}p\, \mathrm{e}^{\mathrm{i}xq + \mathrm{i}yp} f(p,q) \tag{10.6}$$

Thus, the *Weyl quantization*, $f(p,q) \to \hat{f}$, is defined as follows:

$$\hat{f} = \frac{1}{2\pi} \int \mathrm{d}q\,\mathrm{d}p\, \hat{w}(p,q) f(p,q) \tag{10.7}$$

where

$$\hat{w}(p,q) = \frac{1}{2\pi} \int \mathrm{d}x\,\mathrm{d}y\, \hat{e}(x,y)\, \mathrm{e}^{\mathrm{i}xq + \mathrm{i}yp}$$

and is called *the quantization kernel*. The operator $\hat{w}((p,q))$, possesses several remarkable properties. Denoting by $g = (\mu, \nu; z)$ the element of the Heisenberg–Weyl group $H(1)$, we can easily check that $\hat{w}(p,q)$ is,

1. covariant: $T_g\, \hat{w}(p,q) T_g^{\dagger} = \hat{w}(g^{-1} \circ (p,q))$
   where $g^{-1} \circ (p,q) = (p - \mu, q + \nu)$
2. hermitian: $\hat{w}(p,q) = \hat{w}^{\dagger}(p,q)$
3. normalized: $Tr\, \hat{w}(p,q) = 1$
4. trace orthogonal: $Tr\left(\hat{w}(p,q)\hat{w}(p',q')\right) = 2\pi\delta\left(p - p'\right)\delta\left(q - q'\right)$

Note, that the last property allows us to invert Equation 10.7 obtaining,

$$f(p,q) = Tr\left(\hat{w}(p,q)\hat{f}\right) = W_f(p,q)$$

and $W_f(p,q)$ – is called the *symbol* of the operator $\hat{f}$. Thus, the kernel $\hat{w}(p,q)$ establishes an isomorphism between an operator and its symbol

$$\hat{f} \overset{\hat{w}}{\longleftrightarrow} W_f(p,q)$$

It is worth noting that different orderings of the exponential in Equation 10.5 can be used

$$e^{-ixq-iyp} \to \begin{cases} e^{-ix\hat{q}-iy\hat{p}} \\ e^{-ix\hat{q}}\, e^{-iy\hat{p}} \\ e^{-iy\hat{p}}\, e^{-ix\hat{q}} \\ \dots \end{cases}$$

which, as we see below, allows to establish different kinds of correspondence between the operator and its symbol.

10.1.2
Moyal–Stratonovich–Weyl Quantization

The above approach can be generalized into a wide class of physical systems, and is known as *the Moyal–Stratonovich–Weyl quantization* program. Given $(\mathcal{M}, G, \mathcal{H})$, where $\mathcal{M}$ is a symplectic manifold (classical phase space), G is an invariance group of $\mathcal{M}$, and $\mathcal{H}$ is a Hilbert space where an irreducible representation T_g of G is realized, then a linear isomorphism $F^{\infty}(\mathcal{M}) \longleftrightarrow \mathrm{L}(\mathcal{H})$ is established through the quantization kernel $\hat{w}(\xi) \in \mathrm{L}(\mathcal{H})$, $\xi \in \mathcal{M}$ such that

1. $T_g \hat{w}(\xi) T_g^{\dagger} = \hat{w}(g^{-1} \circ \xi),\ g \in G$
2. $\hat{w}(\xi) = \hat{w}(\xi)^{\dagger}$
3. $\mathrm{Tr}\hat{w}(\xi) = 1$
4. $\mathrm{Tr}\left(\hat{w}(\xi)\hat{w}(\xi')\right) = \Delta(\xi, \xi')$

where $\Delta(\xi, \xi')$ is the reproductive kernel:

$$\int_{\mathcal{M}} d\mu(\xi)\Delta(\xi, \xi')g(\xi) = g(\xi') \tag{10.8}$$

and $d\mu(\xi)$ is an invariant measure on $\mathcal{M}$. If such a kernel $\hat{w}(\xi)$ exists, then the quantization procedure is

$$f(\xi) \to \hat{f} = \int_{\mathcal{M}} d\mu(\xi)\hat{w}(\xi)f(\xi) \tag{10.9}$$

where $f(\xi) \in F^{\infty}(\mathcal{M})$ and the inverse procedure (*dequantization*) is

$$\hat{f} \to f(\xi) = \mathrm{Tr}\left(\hat{w}(\xi)\hat{f}\right) \equiv W_f(\xi) \tag{10.10}$$

The function $W_f(\xi)$ is called the *symbol* of $\hat{f}$ and possesses the following properties:

1. $W_{\tilde{f}}(\xi) = W_f(g \circ \xi)$, where $\tilde{f} = T_g \hat{f}\, T_g^{\dagger}$
2. $W_I(\xi) = 1$, where I is the identity operator
3. $W_{f^{\dagger}}(\xi) = W_f^*(\xi)$
4. $Tr(\hat{f}\hat{g}) = \int d\mu(\xi)\, W_f(\xi)\, W_g(\xi)$

In the particular case when $\hat{f}$ is the density matrix ρ, its symbol $W_\rho(\xi) = W(\xi)$ is usually called the *quasi-distribution function*.

10.1.3
Ordering Problem in $L(H)$

Obviously, the (invertible) map $\hat{f} \to f(\xi)$ depends on the ordering rules of functions of noncommutative operators. This can be taken into account by introducing an additional index s, which specifies a certain operator ordering, such that $\hat{f} \to f^{(s)}(\xi)$. It is convenient to introduce the ordering parameter s directly into the kernel $\hat{w}(\xi) \to \hat{w}_s(\xi)$, so that s-ordered symbols $W_f^{(s)}(\xi)$ can be naturally defined.

There are three well-known orderings. $|\xi\rangle$ being the coherent state of G (see Appendix 11.2), then

- $P_f(\xi)$ is the contravariant (Glauber) symbol ($s = 1$), when the correspondence is given in terms of the expansion of an operator over a basis of coherent states:

$$\hat{f} \to P_f(\xi), \quad \hat{f} = \int d\mu(\xi) P_f(\xi) |\xi\rangle\langle\xi| \tag{10.11}$$

- $Q_f(\xi)$ is the (Husimi) covariant symbol ($s = -1$), defined in the opposite sense:

$$\hat{f} \to Q_f(\xi), \quad Q_f(\xi) = \langle\xi|\hat{f}|\xi\rangle \tag{10.12}$$

These two symbols are dual, i.e. they are connected through the following relations:

$$\mathrm{Tr}(\hat{f}\hat{g}) = \int_{\mathcal{M}} d\mu(\xi) P_f(\xi) Q_g(\xi) = \int_{\mathcal{M}} d\mu(\xi) Q_f(\xi) P_g(\xi)$$

It is interesting to note that in this representation, only *diagonal* matrix elements in the basis of coherent states either of the operator $\hat{f}$ or $\hat{g}$ are needed to calculate the corresponding overlap relation.

- Finally, the so-called self-dual (Wigner) symbol, $s = 0$,

  $$\hat{f} \to W_f(\xi)$$

  is defined in such a way that

  $$\mathrm{Tr}(\hat{f}\hat{g}) = \int_{\mathcal{M}} d\mu(\xi) W_f(\xi) W_g(\xi)$$

In general, the correspondence

$$\hat{f} \leftrightarrow W_f^{(s)}(\xi)$$

is realized using the s-ordered kernel $\hat{w}_s(\xi)$ as follows

$$W_f^{(s)}(\xi) = \mathrm{Tr}\left(\hat{f}\,\hat{w}_s(\xi)\right)$$

$$\hat{f} = \int_{\mathcal{M}} \mathrm{d}\mu(\xi)\hat{w}_{-s}(\xi) W_f^{(s)}(\xi)$$

so that the overlap relation has the form

$$\mathrm{Tr}\left(\hat{f}\hat{g}\right) = \int_{\mathcal{M}} d\mu(\xi) W_g^{(s)}(\xi)\, W_f^{(-s)}(\xi) \tag{10.13}$$

which means that the kernel $\hat{w}_s(\xi)$ satisfies the following condition:

$$\mathrm{Tr}\left(\hat{w}_s(\xi)\hat{w}_{-s}(\xi')\right) = \Delta(\xi, \xi') \tag{10.14}$$

where $\Delta(\xi, \xi')$ is a reproductive kernel (Equation 10.8). All the other properties are the same.

10.1.4
Star Product

The cornerstone of Stratonovich–Weyl correspondence is a specific symbol calculus, the so-called star- (or twisted) product $W_f(\xi) * W_g(\xi)$, defined as

$$W_{fg}(\xi) = W_f(\xi) * W_g(\xi) = \hat{L}_{fg}(\xi)\left(W_f(\xi) W_g(\xi)\right) \tag{10.15}$$

where $\hat{L}_{fg}(\xi)$ is a differential or integral operator applied to the product of the symbols. This star product allows us to replace the manipulations with operators in the Hilbert space by differential (or integral) operations in the space of c-valued functions.

It is easy to obtain an integral form of the star-product operation. Using the general expression

$$W_{fg}(\xi) = \mathrm{Tr}\left(\hat{w}(\xi)\hat{f}\hat{g}\right)$$

and Equation 10.9, we immediately get

$$W_f(\xi) * W_g(\xi) = \int_{\mathcal{M}} \mathrm{d}\mu(\xi) K(\xi, \xi_1, \xi_2) W_f(\xi_1) W_g(\xi_2)$$

where

$$K(\xi, \xi_1, \xi_2) = \mathrm{Tr}\left(\hat{w}(\xi)\hat{w}(\xi_1)\hat{w}(\xi_2)\right)$$

The main properties of the star-product operation are

1. Associativity:

$$W_f(\xi) * \left(W_g(\xi) * W_h(\xi)\right) = \left(W_f(\xi) * W_g(\xi)\right) * W_h(\xi)$$

2. Non-commutativity:

$$W_f(\xi) * W_g(\xi) \neq W_g(\xi) * W_f(\xi)$$

10.1.5
Phase–Space Representation and Quantum–Classical Correspondence

Now, we return to our principal task: a representation of quantum systems in the classical phase space. As we have already seen the "dequantization" process

$$\text{Quantum mechanics} \xrightarrow{\hat{w}_s(\xi)} \text{Classical mechanics}$$

consists in the representation of a quantum operator as a c function defined in a manifold $\mathcal{M}$, which is a classical phase space corresponding to a given quantum system

$$\hat{f} \to W_f^{(s)}(\xi)$$

and in a particular case of the density matrix, we get the quasi-distribution function

$$\rho \to W^{(s)}(\xi)$$

The average values are computed according to Equation 10.13, i.e. as an integral of the s-ordered symbol with the $(-s)$-ordered quasi-distribution function

$$\langle \hat{f} \rangle = \mathrm{Tr}\left(\hat{f}\rho\right) = \int_{\mathcal{M}} \mathrm{d}\mu(\xi)\, W_f^{(s)}(\xi)\, W^{(-s)}(\xi) \tag{10.16}$$

The star-product operation allows us to rewrite the Schrdinger equation for the density matrix

$$i\hbar\partial_t \rho = [H, \rho]$$

where H is the system Hamiltonian, as a Liouville-type equation for the quasi-distribution function. Introducing the Moyal brackets as a symbol of the commutator

$$[\hat{f}, \hat{g}] \to \{W_f^{(s)}, W_g^{(s)}\}_M = W_f^{(s)} * W_g^{(s)} - W_g^{(s)} * W_f^{(s)}$$

we obtain the following classical-like evolution equation for $W^{(s)}(\xi)$:

$$i\hbar\partial_t W^{(s)}(\xi) = \{W_H^{(s)}(\xi), W^{(s)}(\xi)\}_M$$

In this book, we are interested in applications of quasi-classical methods only to two types of quantum systems: two-level atoms and quantized electromagnetic fields. In both cases, we can introduce the so-called semiclassical expansion parameter ϵ, which indicates how close a quantum state is to the corresponding "classical" state, so that the transition from quantum to classical mechanics corresponds to the limit $\epsilon \to 0$. Of course, this condition is necessary but not sufficient, since only a certain class of quantum states allows a smooth transition to the classical mechanics limit.

In the particular case of a quantized field mode, the semiclassical parameter is the inverse average number of photons, while for the atomic system, it is the inverse total number of atoms. As is discussed below, it turns out that in both quantum systems the self-dual symbols (corresponding to $s = 0$) possess a very important dynamic property, namely, in the semiclassical limit, the evolution equations for the quasi-distribution function do not contain the second-order terms in the expansion of the Liouville equation in the semiclassical parameter ϵ. As a result, the evolution equation takes the following form in appropriate variables:

$$i\partial_t W^{(0)}(\xi) = \epsilon\{W_H^{(0)}(\xi), W^{(0)}(\xi)\} + O(\epsilon^3) \tag{10.17}$$

with the corresponding solution

$$W^{(0)}(\xi, t) \approx W^{(0)}(\xi(t), t = 0) \tag{10.18}$$

where $\xi(t)$ are the classical trajectories in the phase space. The above equation means that, in the leading order in inverse powers of the small parameter ϵ, each point of the initial (quantum) distribution evolves along classical trajectories on the corresponding symplectic manifold.

Usually, there exists a class of initial states, (the so-called semiclassical states, for which this approximation describes well the initial stage of quantum dynamics (for times when one can neglect the self-interference) even for the cases where the Hamiltonian is a nonlinear form on the generators of the system dynamic symmetry group. These states are specified by the interaction Hamiltonian in such a way that the norm of the operator describing quantum corrections in Equation 10.17 on these states is small. Usually, in physical applications, semiclassical states have the form of localized states (for instance, coherent-like states).

The semiclassical solution (10.18) allows us to calculate the mean values of the operators corresponding to observables of a given quantum system leading to drastically better results than the "naive" solution of the Heisenberg equations of motion with decoupled correlators. $\hat{f}$ being the element of the algebra of observables $\{\hat{f}_j, j = 1, 2, \ldots\}$ corresponding to a given quantum system, its average value evolution can be computed as follows (Equation 10.16):

$$\langle \hat{f}_j(t)\rangle = \int_{\mathcal{M}} \mathrm{d}\mu(\xi)\, W_{f_j}^{(0)}(\xi)\, W^{(0)}(\xi, t) \tag{10.19}$$

where $W^{(0)}(\xi, t)$ – is the solution of Equation10.17 and $W_{f_j}^{(0)}(\xi)$ is the Wigner symbol of $\hat{f}_j$. Differentiating both sides of the above equation with respect to time t, and using Equation 10.16, one obtains

$$i\partial_t\langle \hat{f}_j(t)\rangle = \epsilon \int_{\mathcal{M}} \mathrm{d}\mu(\xi)\, W_{f_j}^{(0)}(\xi, t)\{W_H^{(0)}(\xi), W^{(0)}(\xi, t)\}$$

where $W_H^{(0)}(\xi)$ is the Wigner symbol of the Hamiltonian. Making use of the Leibniz rule, $\{f, \{g, h\}\} = g\{f, h\} + \{f, g\}h$ the above equation is transformed into the form

$$i\partial_t\langle \hat{f}_j(t)\rangle = \epsilon \int_{\mathcal{M}} \mathrm{d}\mu(\xi)\, W^{(0)}(\xi, t)\{W_{f_j}^{(0)}(\xi), W_H^{(0)}(\xi)\}$$

The Poisson brackets can be seen as a symbol of some (in general, nonlinear) function F of elements of the algebra of observables, $\epsilon\{W_{f_j}^{(0)}(\xi), W_H^{(0)}(\xi)\} = \sum_k \alpha_{jk} W_{F(\hat{f}_k)}(\xi)$. Thus, one can write

$$i\partial_t\langle\hat{f}_j(t)\rangle = \int_{\mathcal{M}} \mathrm{d}\mu(\xi) W(\xi,t) \sum_k \alpha_{jk} W_{F(\hat{f}_k)}(\xi) = \sum_k \alpha_{jk}\langle F(\hat{f}_k)(t)\rangle \tag{10.20}$$

On the other hand, the averaged Heisenberg equations can be written down in the form similar to Equation 10.20:

$$\frac{\mathrm{d}}{\mathrm{d}t}\partial_t\langle\hat{f}_j(t)\rangle = \frac{1}{i\hbar}\langle[\hat{f}_j(t), H]\rangle = \sum_k \beta_{jk}\langle F(\hat{f}_k)(t)\rangle \tag{10.21}$$

and according to Equation 10.17 $\alpha_{jk} \to \beta_{jk} + O(\epsilon^2)$, when $\epsilon \to 0$. This allows us to use the semiclassical solution (10.18) for describing phenomena such as squeezing and entanglement of certain quantum states.

It is interesting to note that the quasi-classical evolution equation for the Wigner function does not lead to the classical equations of motion for the average values of the system's operators

$$\partial_t\langle\hat{f}_j(t)\rangle = \sum_k \beta_{jk} F(\langle\hat{f}_k(t)\rangle) \tag{10.22}$$

where β_{jk} are the same as in Equation 10.21, in the sense that we do not arrive at equations with decoupled correlators. The solutions of Equations 10.22 are usually called *parametric approximations*. It is well known that such a ''classical'' solution diverges from the exact one even for very small times for practically any initial condition.

This means that even in the semiclassical limit some quantum features are preserved. On the other hand, the quantum phenomena that are consequences of the self-interference effect (like Schrödinger cats) are clearly beyond the scope of such semiclassical approximation.

10.2
Atomic Quasi-distributions

Now we proceed with the discussion of some types of atomic quasi-distributions. Instead of the atomic basis (Dicke basis) $|k, A\rangle, k = 0, \ldots, A$ (Equation 1.9), we use the standard angular momentum basis $|m, S\rangle$, $m = -S, \ldots, S$, $S = A/2$; the relation between two bases is

$$|m, S\rangle = |k = m + S, A = 2S\rangle \tag{10.23}$$

$$|k, A\rangle = |m = k - A/2, S = A/2\rangle \tag{10.24}$$

so that

$$S_z|m, S\rangle = m|m, S\rangle$$

In particular, the coherent atomic states (1.17) in the angular momentum basis (10.23) have the form

$$|\theta,\phi\rangle = \mathrm{e}^{-\mathrm{i}\varphi S_z}\mathrm{e}^{-\mathrm{i}\theta S_Y}|S,S\rangle \tag{10.25}$$

$$|\theta,\phi\rangle = \sum_{m=-S}^{S}\sqrt{\frac{(2S)!}{(S+m)!(S-m)!}}\,\mathrm{e}^{-im\phi}\cos^{S+m}\frac{\theta}{2}\sin^{S-m}\frac{\theta}{2}\,|m,S\rangle$$

We start by considering separately different types of atomic quasi-distributions. After that we discuss a general case in the *s*-parameterized form.

10.2.1
P Function

The P function for an angular momentum is introduced as the decomposition coefficient of the density matrix in the coherent state basis

$$\rho = \frac{2S+1}{4\pi}\int \mathrm{d}\Omega|\mathbf{n}\rangle\langle\mathbf{n}|P(\mathbf{n}) \tag{10.26}$$

where $|\mathbf{n}\rangle = |\theta,\phi\rangle$ and $\mathbf{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ indicate the point on the sphere. It is easy to see that due to the normalization condition $\mathrm{Tr}\rho = 1$ the following relation is satisfied:

$$\frac{2S+1}{4\pi}\int \mathrm{d}\Omega P(\mathbf{n}) = 1$$

To invert Equation 10.26 we multiply it by the irreducible tensor operator (ITO) $\hat{T}^{(S)\dagger}_{L,M}$ (see Appendix 11.6) and take the trace, obtaining

$$\mathrm{Tr}\left(\hat{T}^{(S)\dagger}_{L,M}\rho\right) = \frac{2S+1}{4\pi}\int \mathrm{d}\Omega\langle\mathbf{n}|\hat{T}^{(S)\dagger}_{L,M}|\mathbf{n}\rangle P(\mathbf{n})$$

Matrix elements of the ITO (Equation 11.57) have the form

$$\langle\mathbf{n}|\,\hat{T}^{(S)\dagger}_{L,M}|\mathbf{n}\rangle = \sqrt{\frac{4\pi}{2S+1}}C^{S,S}_{S,S;L,0}Y^{*}_{LM}(\mathbf{n})$$

where $Y_{LM}(\mathbf{n})$ are spherical harmonics and $C^{S,m'}_{S,m;L,M}$ are the Clebsch–Gordan coefficients. We the obtain the following Relation:

$$\sqrt{\frac{4\pi}{2S+1}}\mathrm{Tr}\left(\hat{T}^{(S)\dagger}_{L,M}\rho\right)\left(C^{S,S}_{S,S;L,0}\right)^{-1} = \int \mathrm{d}\Omega\; Y^{*}_{LM}(\mathbf{n})P(\mathbf{n})$$

Multiplying by $Y_{LM}(\mathbf{n}')$ and summing over L, M, we obtain owing to Equation 11.46, the following expression:

$$P(\mathbf{n}) = \sqrt{\frac{4\pi}{2S+1}}\sum_{L=0}^{2S}\sum_{M=-L}^{L}\mathrm{Tr}\left(\hat{T}^{(S)\dagger}_{L,M}\rho\right)Y_{LM}(\mathbf{n})\left(C^{S,S}_{S,S;L,0}\right)^{-1} \tag{10.27}$$

which allows us to find the function $P(\mathbf{n})$. We can represent Equation 10.27 as

$$P(\mathbf{n}) = \mathrm{Tr}\left(\rho \hat{w}_P(\mathbf{n})\right) \tag{10.28}$$

where the kernel

$$\hat{w}_P(\mathbf{n}) = \sqrt{\frac{4\pi}{2S+1}} \sum_{L=0}^{2S} \sum_{M=-L}^{L} \hat{T}_{L,M}^{(S)\dagger} Y_{LM}(\mathbf{n}) \left(C_{S,S;L,0}^{S,S}\right)^{-1} \tag{10.29}$$

is a normalized Hermitian operator, which satisfies

$$\hat{w}_P(\mathbf{n}) = \hat{w}_P^{\dagger}(\mathbf{n}), \quad \mathrm{Tr}\hat{w}_P(\mathbf{n}) = 1, \quad \frac{2S+1}{4\pi} \int d\Omega \hat{w}_P(\mathbf{n}) = I \tag{10.30}$$

Using the expansion (Equation 11.52)

$$e^{-i\omega \mathbf{n}\cdot\mathbf{S}} = \sqrt{\frac{4\pi}{2S+1}} \sum_{L=0}^{2S} \sum_{M=-L}^{L} (-i)^L \chi_L^S(\omega)\, Y_{L,M}^{*}(\mathbf{n})\, \widehat{T}_{L,M}^{(S)} \tag{10.31}$$

where $\mathbf{S} = (S_x, S_y, S_z)$, and the orthogonality relation (11.54) for the generalized characters (Equation 11.53) $\chi_L^S(\omega)$, we represent the operator (10.29) in the integral form

$$\hat{w}_P(\theta, \phi) = \int_0^{2\pi} d\omega\, e^{i\omega\, \mathbf{n}\cdot\mathbf{S}} \kappa^{(1)}(\omega) \tag{10.32}$$

where the function $\kappa^{(1)}(\omega)$ is defined as

$$\kappa^{(1)}(\omega) = \frac{1}{2\pi} \sum_{L=0}^{2S} (-i)^L \frac{2L+1}{2S+1} \chi_L^S(\omega) \left(C_{S,S;L,0}^{S,S}\right)^{-1}$$

Obviously, we can introduce the P symbol for any operator $\hat{f}$ in the same way as the $P(\mathbf{n})$ function is defined:

$$\hat{f} = \frac{2S+1}{4\pi} \int d\Omega |\mathbf{n}\rangle\langle\mathbf{n}| P_f(\mathbf{n})$$

10.2.2
Q Function

The Q function for an angular momentum is defined as the average value of the density matrix in the coherent state $|\mathbf{n}\rangle$

$$Q(\mathbf{n}) = \langle \mathbf{n}|\rho|\mathbf{n}\rangle \tag{10.33}$$

and likewise, the Q-symbol for any operator $\hat{f}$ is

$$Q_f(\mathbf{n}) = \langle \mathbf{n}|\hat{f}|\mathbf{n}\rangle$$

It is easy to see that the average of an operator $\hat{f}$ in a state with the density matrix $\boldsymbol{\rho}$ can be calculated in two ways

$$\langle \hat{f} \rangle = \mathrm{Tr}\left(\hat{f}\rho\right) = \frac{2S+1}{4\pi}\int d\Omega Q_{\rho}(\mathbf{n})P_{f}(\mathbf{n}) = \frac{2S+1}{4\pi}\int d\Omega Q_{f}(\mathbf{n})P_{\rho}(\mathbf{n})$$

which are particular cases of the so-called overlap relation

$$\mathrm{Tr}\left(\hat{f}\hat{g}\right) = \frac{2S+1}{4\pi}\int d\Omega Q_{f}(\mathbf{n})P_{g}(\mathbf{n}) = \frac{2S+1}{4\pi}\int d\Omega P_{f}(\mathbf{n})Q_{g}(\mathbf{n}) \tag{10.34}$$

where $Q_{f,g}(\mathbf{n})$ and $P_{f,g}(\mathbf{n})$ are Q and P symbols of the $\hat{f}$ and $\hat{g}$ operators.

The density matrix ρ can be expanded in the basis of the irreducible tensor operators $\hat{T}^{(S)}_{L,M}$ (see Appendix 11.5) as

$$\rho = \sum_{L=0}^{2S}\sum_{M=-L}^{L}\hat{T}^{(S)}_{L,M}\mathrm{Tr}\left(\hat{T}^{(S)\dagger}_{L,M}\rho\right) \tag{10.35}$$

Substituting Equation 10.35 in Equation 10.33 and taking into account Equation 11.57, we obtain

$$Q(\mathbf{n}) = \sqrt{\frac{4\pi}{2S+1}}\sum_{L=0}^{2S}\sum_{M=-L}^{L}\mathrm{Tr}\left(\hat{T}^{(S)\dagger}_{L,M}\rho\right)Y_{LM}(\mathbf{n})C^{S,S}_{S,S;L,0} \tag{10.36}$$

Owing to the properties of the spherical harmonics (Equation 11.43), it can be easily seen that the following normalization condition is satisfied:

$$\frac{2S+1}{4\pi}\int d\Omega Q(\mathbf{n}) = 1 \tag{10.37}$$

The $Q(\mathbf{n})$ function can be represented in a manner similar to Equation 10.28

$$Q(\mathbf{n}) = \mathrm{Tr}\left(\hat{w}_{Q}\rho\right) \tag{10.38}$$

where the operator $\hat{w}_Q$ is now defined as

$$\hat{w}_{Q}(\mathbf{n}) = |\mathbf{n}\rangle\langle\mathbf{n}| = \sqrt{\frac{4\pi}{2S+1}}\sum_{L=0}^{2S}\sum_{M=-L}^{L}\hat{T}^{(S)\dagger}_{L,M}Y_{LM}(\mathbf{n})C^{S,S}_{S,S;L,0} \tag{10.39}$$

such that

$$\hat{w}_{Q}(\mathbf{n}) = \hat{w}^{\dagger}_{Q}(\mathbf{n}), \quad \mathrm{Tr}\hat{w}_{Q}(\mathbf{n}) = 1, \quad \frac{2S+1}{4\pi}\int d\Omega\hat{w}_{Q}(\mathbf{n}) = I \tag{10.40}$$

Comparing Equations 10.29 and 10.39, we observe that the operators $\hat{w}_Q(\mathbf{n})$ and $\hat{w}_P(\mathbf{n})$ are dual in the sense that

$$\mathrm{Tr}\left(\hat{w}_{Q}(\mathbf{n})\hat{w}_{P}(\mathbf{n}')\right) = \frac{4\pi}{2S+1}\delta(\mathbf{n},\mathbf{n}'), \quad \mathbf{n},\mathbf{n}' \in S^2 \tag{10.41}$$

where $\delta(\mathbf{n},\mathbf{n}')$ is the Dirac delta on the sphere,

$$\int d\Omega f(\mathbf{n})\delta(\mathbf{n},\mathbf{n}') = f(\mathbf{n}')$$

Here $f(\mathbf{n})$ is an arbitrary function on the sphere (see Equation 11.46). Also, (see Equation 11.56) the following orthogonality relations are satisfied:

$$\frac{2S+1}{4\pi}\int d\Omega\,\left(\hat{w}_Q(\mathbf{n})\right)_{mm'}\left(\hat{w}_P(\mathbf{n})\right)_{kk'} = \delta_{mk'}\,\delta_{m'k} \tag{10.42}$$

which is equivalent to the overlap relation (10.34).

Equation 10.36 can be inverted. Using the orthogonality of spherical harmonics (Equation 11.42), we obtain from Equation 10.36 (by multiplying by $Y^*_{LM}(\mathbf{n})$ and integrating over the sphere)

$$\mathrm{Tr}\left(\hat{T}^{(S)\dagger}_{L,M}\,\rho\right) = \sqrt{\frac{2S+1}{4\pi}}\left(C^{S,S}_{S,S;L,0}\right)^{-1}\int d\Omega\,Y^*_{LM}(\mathbf{n})Q(\mathbf{n})$$

Substituting the above equation into Equation 10.35, the formula known as the *tomographic reconstruction relation* is obtained, which allows us to reconstruct all the elements of the density matrix of an angular momentum system by simply measuring the corresponding $Q(\mathbf{n})$ function (the diagonal elements in the basis of the coherent states) [111, 112],

$$\rho = \sqrt{\frac{2S+1}{4\pi}}\sum_{L=0}^{2S}\sum_{M=-L}^{L}\left(C^{S,S}_{S,S;L,0}\right)^{-1}\hat{T}^{(S)}_{L,M}\int d\Omega\,Y^*_{LM}(\mathbf{n})Q(\mathbf{n}) \tag{10.43}$$

The operator $\hat{w}_Q(\mathbf{n})$ can be represented in the integral form in a way similar to Equation 10.32,

$$\hat{w}_Q(\mathbf{n}) = \int_0^{2\pi} d\omega\,e^{i\omega\,\mathbf{n}\cdot\mathbf{S}}\kappa^{(-1)}(\omega) \tag{10.44}$$

where, in this case, the function $\kappa^{(-1)}(\omega)$ has a very simple form

$$\kappa^{(-1)}(\omega) = \frac{1}{2\pi}\sum_{L=0}^{2S}(-i)^L\frac{2L+1}{2S+1}\chi^S_L(\omega)C^{S,S}_{S,S;L,0} = \frac{1}{2\pi}\,e^{-i\omega S} \tag{10.45}$$

Thus, Equation 10.44 acquires a simple form

$$\hat{w}_Q(\mathbf{n}) = \frac{1}{2\pi}\int_0^{2\pi} d\omega\,e^{i\omega\,(\mathbf{n}\cdot\mathbf{S}-S)} \tag{10.46}$$

and the function $Q(\mathbf{n})$ is represented as a Fourier transform

$$Q(\mathbf{n}) = \frac{1}{2\pi}\int_0^{2\pi} d\omega\mathrm{Tr}\left(e^{i\omega\,\mathbf{n}\cdot\mathbf{S}}\rho\right)e^{-i\omega\,S} \tag{10.47}$$

The inversion formula (10.43) can also be written as

$$\rho = \int d\Omega\int_0^{2\pi} d\omega\,e^{-i\omega\,\mathbf{n}\cdot\mathbf{S}}g(\omega)Q(\mathbf{n}) \tag{10.48}$$

where

$$g(\omega) = \frac{1}{8\pi^2}\sum_{L=0}^{2S} i^L\,(2L+1)\,\chi^S_L(\omega)\left[C^{S,k}_{S,k;L,0}\right]^{-1}$$

is a function that does not depend on the system characteristics.

10.2.3
Stratonovich–Weyl Distribution

The self-dual kernel is defined to satisfy the following overlap relations (compare with Equations 10.41 and 10.42):

$$\mathrm{Tr}\left(\hat{w}_S(\mathbf{n})\hat{w}_S(\mathbf{n}')\right) = \frac{4\pi}{2S+1}\delta(\mathbf{n},\mathbf{n}') \tag{10.49}$$

and

$$\frac{2S+1}{4\pi}\int \mathrm{d}\Omega \left(\hat{w}_S(\mathbf{n})\right)_{mm'}\left(\hat{w}_S(\mathbf{n})\right)_{kk'} = \delta_{mk'}\delta_{m'k} \tag{10.50}$$

The Wigner (or also Stratonovich–Weyl) distribution [110, 113] (see also [18]) is defined as

$$W(\mathbf{n}) = \mathrm{Tr}\left(\rho\hat{w}_S(\mathbf{n})\right) \tag{10.51}$$

where

$$\hat{w}_S(\mathbf{n}) = \sqrt{\frac{4\pi}{2S+1}}\sum_{L=0}^{2S}\sum_{M=-L}^{L}\hat{T}_{L,M}^{(S)\dagger}Y_{LM}(\mathbf{n}) \tag{10.52}$$

It can be easily seen that the operator $\hat{w}_S(\mathbf{n})$ has the same properties as the operators $\hat{w}_P(\mathbf{n})$ and $\hat{w}_Q(\mathbf{n})$ (10.30), (10.40):

$$\hat{w}_S(\mathbf{n}) = \hat{w}_S^{\dagger}(\mathbf{n}), \quad \mathrm{Tr}\hat{w}_S(\mathbf{n}) = 1, \quad \frac{2S+1}{4\pi}\int \mathrm{d}\Omega\hat{w}_S(\mathbf{n}) = I \tag{10.53}$$

which imply that the quasi-distribution $W(\mathbf{n})$ is real and normalized according to

$$\frac{2S+1}{4\pi}\int \mathrm{d}\Omega W(\mathbf{n}) = 1$$

The overlap relations (Equations 10.49 to 10.50) determine the procedure for calculation of the average values

$$\langle\hat{f}\rangle = \mathrm{Tr}\left(\hat{f}\rho\right) = \frac{2S+1}{4\pi}\int \mathrm{d}\Omega W_f(\mathbf{n})W_w(\mathbf{n})$$

where $W_f(\mathbf{n})$ is the W-symbol of the operator $\hat{f}$, and, in general, we have

$$\mathrm{Tr}\left(\hat{f}\hat{g}\right) = \frac{2S+1}{4\pi}\int \mathrm{d}\Omega W_f(\mathbf{n})W_g(\mathbf{n})$$

The inversion formula for Equation 10.51 is obtained in the same way as for Equation 10.43:

$$\rho = \sqrt{\frac{2S+1}{4\pi}}\sum_{L=0}^{2S}\sum_{M=-L}^{L}\hat{T}_{L,M}^{(S)}\int \mathrm{d}\Omega\, Y_{LM}^{*}(\mathbf{n})W(\mathbf{n}) \tag{10.54}$$

The following is an integral representation for the Wigner operator $\hat{w}_S(\mathbf{n})$, [114],

$$\hat{w}_S(\mathbf{n}) = \int_0^{2\pi} \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega\, \mathbf{n}\cdot\mathbf{S}} \kappa^{(0)}(\omega) \tag{10.55}$$

where

$$\kappa^{(0)}(\omega) = \frac{1}{2\pi} \sum_{L=0}^{2S} (-i)^L \frac{2L+1}{2S+1} \chi_L^S(\omega) \tag{10.56}$$

One can show that the function $\kappa^{(0)}(\omega)$ in the limiting case of large spin (large number of atoms), $S \gg 1$, has the asymptotic form

$$\kappa^{(0)}(\omega) \to (-1)^S \left[\delta(\omega - \pi) - \frac{i}{S} \frac{\partial}{\partial\omega} \delta(\omega - \pi) \right] \tag{10.57}$$

where the limit is understood in a weak sense.

10.2.4
s-Ordered Distributions

The operators $\hat{w}_P(\mathbf{n})$, $\hat{w}_Q(\mathbf{n})$, and $\hat{w}_S(\mathbf{n})$ can be written in the following way:

$$\hat{w}^{(s)}(\mathbf{n}) = \sqrt{\frac{4\pi}{2S+1}} \sum_{L=0}^{2S} \sum_{M=-L}^{L} \hat{T}_{L,M}^{(S)\dagger} Y_{LM}(\mathbf{n}) \left(C_{SS,L0}^{SS} \right)^{-s}$$

where different operators correspond to the different values of the parameter s

$$\hat{w}^{(1)}(\mathbf{n}) = \hat{w}_P(\mathbf{n}), \quad \hat{w}^{(-1)}(\mathbf{n}) = \hat{w}_Q(\mathbf{n}), \quad \hat{w}^{(0)}(\mathbf{n}) = \hat{w}_S(\mathbf{n})$$

One can obviously write

$$\hat{w}^{(s)}(\mathbf{n}) = D(\theta, \varphi) T_z^{(s)} D^\dagger(\theta, \varphi) \tag{10.58}$$

where $D(\theta, \phi) = \exp\left[-\frac{\theta}{2}\left(S_+ \mathrm{e}^{-\mathrm{i}\phi} - S_- \mathrm{e}^{\mathrm{i}\phi}\right)\right]$ is the displacement operator on the sphere (see Appendix 11.2) and the diagonal operator $T_z^{(s)}$ is defined by its integral representation

$$T_z^{(s)} = \int_0^{2\pi} \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega\, S_z} \kappa^{(s)}(\omega) \tag{10.59}$$

$$\kappa^{(s)}(\omega) = \frac{1}{2\pi} \sum_{L=0}^{2S} (-i)^L \frac{2L+1}{2S+1} \chi_L^S(\omega) \left(C_{S,S;L,0}^{S,S} \right)^{-s} \tag{10.60}$$

The operators $\hat{w}^{(s)}(\mathbf{n})$ are covariant with respect to $SU(2)$ rotations, that is, the following is satisfied:

$$T(g)\hat{w}^{(s)}(\mathbf{n})T^{-1}(g) = \hat{w}^{(s)}(g^{-1}\circ\mathbf{n}), \quad g \in SU(2) \tag{10.61}$$

where $T(g)$ are operators of the $2S+1$-dimensional unitary irreducible representation of the $SU(2)$ group. Indeed, from Equations 11.45 and 11.49, we obtain (see Appendix 11.6)

$$\begin{aligned} T(g)\hat{w}^{(s)}(\mathbf{n})T^{-1}(g) &= \sqrt{\frac{4\pi}{2S+1}}\sum_{L=0}^{2S}\sum_{M,K=-L}^{L} D_{KM}^{(L)*}(g)\,\hat{T}_{L,K}^{(S)\dagger}\,Y_{LM}(\mathbf{n})\left(C_{SS,L0}^{SS}\right)^{-s} \\ &= \sqrt{\frac{4\pi}{2S+1}}\sum_{L=0}^{2S}\sum_{K=-L}^{L} \hat{T}_{L,K}^{(S)\dagger}\,Y_{LK}(g^{-1}\circ\mathbf{n})\left(C_{SS,L0}^{SS}\right)^{-s} \\ &= \hat{w}^{(s)}(g^{-1}\circ\mathbf{n}) \end{aligned}$$

which implies the covariance of the s-ordered quasi-distributions $W^{(s)}(\mathbf{n})$:

$$\tilde{W}^{(s)}(\mathbf{n}) = W^{(s)}(g\circ\mathbf{n})$$

Here, $\tilde{W}^{(s)}(\mathbf{n})$ is the quasi-distribution corresponding to the transformed density matrix $\tilde{\rho} = T^{-1}(g)\rho T(g)$.

In terms of coefficients of expansion of an arbitrary operator $\hat{f}$ in the basis of ITO

$$\hat{f} = \sum_{L=0}^{2S}\sum_{M=-L}^{L} f_{LM}\hat{T}_{LM}^{(S)}$$

the s-parameterized symbol has a simple form

$$W_f^{(s)}(\theta,\phi) = \frac{2\sqrt{\pi}}{\sqrt{2S+1}}\sum_{L,M}\left(C_{SS,L0}^{SS}\right)^{-s} f_{LM}Y_{LM}(\theta,\phi) \tag{10.62}$$

For instance, the s-parameterized symbols of $su(2)$ algebra generators are

$$W_{S_z}^{(s)}(\theta,\phi) = \left(\frac{S}{S+1}\right)^{-s/2}\sqrt{S(S+1)}\cos\theta \tag{10.63}$$

$$W_{S_\pm}^{(s)}(\theta,\phi) = \left(\frac{S}{S+1}\right)^{-s/2}\sqrt{S(S+1)}\sin\theta\,\mathrm{e}^{\pm i\phi} \tag{10.64}$$

10.2.5
Star Product

The differential form of the s-ordered star product $*_s$ (10.15) $\hat{L}_{fg}(\Omega)$ is given by the expression [115, 116]:

$$W_f^{(s_1)} *_s W_g^{(s_2)} = \hat{L}_{fg}^{(s)}\left(W_f^{(s_1)}W_g^{(s_2)}\right) \tag{10.65}$$

$$\hat{L}_{fg}^{(s)}(\theta,\phi) = N_S\sum_j a_j\,\tilde{F}^{s-1}(\mathcal{L}^2)\left[\left(S^{+(j)}\hat{F}^{1-s_1}(\mathcal{L}^2)\right)_f\otimes\left(S^{-(j)}\hat{F}^{1-s_2}(\mathcal{L}^2)\right)_g\right]$$

where we use the notation $\hat{A}_f \otimes \hat{B}_g(W_f W_g) = (\hat{A}W_f)(\hat{B}W_g)$, and

$$N_S = \sqrt{2S+1}\,[(2S+1)!\,(2S)!]^{s/2}, \quad a_j = \frac{(-1)^j}{j!\,(2S+j+1)!} \tag{10.66}$$

Here, $\mathcal{L}^2$ is a Casimir operator on the sphere

$$\mathcal{L}^2 = -\left[\partial_\theta^2 + \cot\theta\,\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2\right] \tag{10.67}$$

such that

$$\hat{F}(\mathcal{L}^2)\,Y_{L,M}(\theta,\phi) = F(L)\,Y_{L,M}(\theta,\phi) \tag{10.68}$$

with

$$F(L) = \sqrt{(2S+L+1)!\,(2S-L)!} \tag{10.69}$$

In other words, $\hat{F} = \sum_{L,M} F(L)|L,M\rangle\langle L,M|$. The symbolic powers $S^{\pm(j)}$ are introduced according to

$$S^{\pm(j)} = \Pi_{k=0}^{j-1}\left(k\cot\theta - \partial_\theta \mp \frac{i}{\sin\theta}\partial_\phi\right)$$

The number of terms in the sum (10.65) is finite because the corresponding phase space for the $SU(2)$ group is bounded; more precisely,

$$S^{\pm(j)}\,W_f^{(s)}(\theta,\phi) = 0, \quad j > deg\,\hat{f} \tag{10.70}$$

where the degree of nonlinearity, deg $\hat{f}$, (in the generators of the $su(2)$ algebra) of an operator $\hat{f}$, is defined by the maximum value of l, such that $f_{lk} \neq 0$.

It is worth noting that although the operator function $\hat{F}(\mathcal{L}^2)$ enters explicitly into Equation 10.65, nevertheless, the property (10.68) is sufficient to determine the action of $\hat{L}_{fg}(\theta,\phi)$ on the product of any pair of symbols $W_f^{(s_1)}$ and $W_g^{(s_2)}$.

One can observe from Equation 10.65 that the star product acquires the simplest form for the P-symbol, $P_f(\theta,\phi) = W_f^{(s=1)}(\theta,\phi)$,

$$P_{fg}(\theta,\phi) = (2S+1)!\sum_j a_j\left(S^{+(j)}P_f(\theta,\phi)\right)\left(S^{-(j)}P_g(\theta,\phi)\right)$$

where a_j is defined in Equation 10.66. For the Q- symbol, $Q_{fg}(\theta,\phi) = W_f^{(s=-1)}(\theta,\phi)$ and the Stratonovich–Wigner W-symbol, $W_f(\theta,\phi) = W_f^{(s=0)}(\theta,\phi)$, we have, correspondingly,

$$Q_{fg}(\theta,\phi) = \frac{1}{(2S)!}\sum_j a_j^{-2}\hat{F}(\mathcal{L}^2)\left(S^{+(j)}\hat{F}^2(\mathcal{L}^2)Q_f(\theta,\phi)\right)\left(S^{-(j)}\hat{F}^2(\mathcal{L}^2)Q_g(\theta,\phi)\right)$$

$$W_{fg}(\theta,\phi) = \sqrt{2S+1}\sum_j a_j\,\hat{F}^{-1}(\mathcal{L}^2)\left(S^{+(j)}\hat{F}(\mathcal{L}^2)W_f(\theta,\phi)\right)\left(S^{-(j)}\hat{F}(\mathcal{L}^2)W_g(\theta,\phi)\right)$$

Various mixed relations are also possible if we choose different values for the parameter s parameter in Equation 10.65; for example, we can compute the Wigner symbol $W_{fg}(\theta,\phi)$ starting from the Q symbols $Q_f(\theta,\phi)$ and $Q_g(\theta,\phi)$.

To show how the expression (10.65) works, we calculate the s-ordered symbol for the operator S_z^2 starting with the symbol (10.63). First, we note that $W^{(s)}_{S_z}(\theta,\phi) \sim Y_{1m}(\theta,\phi)$, $m = 0, \pm 1$, and hence, $\hat{F}(\mathcal{L}^2)Y_{1m}(\theta,\phi) = F(1)Y_{1m}(\theta,\phi)$. Owing to the property (10.70) only two terms ($j = 0, 1$) contribute to the sum in Equation 10.65:

$$W^{(s_1)}_{S_z} * W^{(s_2)}_{S_z} = N_S F^{2-s_1-s_2}(1)\tilde{F}^{s-1}(\mathcal{L}^2)\left[\frac{1}{(2S+1)!}\, W^{(s_1)}_{S_z} W^{(s_2)}_{S_z} - \frac{1}{(2S+2)!} S_+^{(1)} W^{(s_1)}_{S_z} S_-^{(1)} W^{(s_2)}_{S_z}\right] \tag{10.71}$$

where

$$S^{\pm(1)} = -\frac{\partial}{\partial\theta} \mp \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} \tag{10.72}$$

Substituting Equation 10.63 into Equation 10.71, we get after some algebraic simplification,

$$W^{(s_1)}_{S_z} *_s W^{(s_2)}_{S_z} = N_S F^{2-s_1-s_2}(1)\tilde{F}^{s-1}(\mathcal{L}^2)\left(\frac{S}{S+1}\right)^{-\frac{s_1+s_2}{2}} \times \frac{S\,(S+1)}{(2S+2)}\left[(2S+3)\left(\cos^2\theta - \frac{1}{3}\right) + \frac{2S}{3}\right]$$

and because $\cos^2\theta - \frac{1}{3} \sim Y_{20}(\theta,\phi)$, we obtain

$$W^{(s_1)}_{S_z} *_s W^{(s_2)}_{S_z} = N_S F^{2-s_1-s_2}(1)\left(\frac{S}{S+1}\right)^{-\frac{s_1+s_2}{2}} \frac{S\,(S+1)}{(2S+2)!} \times \left[F^{s-1}(2)\,(2S+3)\left(\cos^2\theta - \frac{1}{3}\right) + F^{s-1}(0)\frac{2S}{3}\right]$$

where

$$F(2) = \sqrt{(2S-2)!(2S+3)!}$$

$$F(1) = \sqrt{(2S-1)!(2S+2)!}$$

$$F(0) = \sqrt{(2S)!(2S+1)!}$$

Finally, we arrive at the following expression for $W^{(s)}_{S_z^2}$:

$$W^{(s)}_{S_z^2}(\theta,\phi) = \frac{1}{2}(S(2S-1))^{(1-s)/2}((2S+3)(S+1))^{(1+s)/2}\left(\cos^2\theta - 1/3\right) + \frac{S\,(S+1)}{3} \tag{10.73}$$

Correspondence Rules

The explicit form of the star-product also allows us to find the *correspondence rules* [116] (see also [117]):

$$\left.\begin{array}{l} \rho S_z \\ \\ S_z \rho \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} \cdot \\ \left(\mp l_z/2 + \Lambda_0^{(s)}(\phi, \theta)\right) W^{(s)}(\theta, \phi) \\ \cdot \end{array}\right. \tag{10.74}$$

$$\left.\begin{array}{l} \rho S_\pm \\ \\ S_\pm \rho \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} \cdot \\ \left(\mp l_\pm/2 + \Lambda_\pm^{(s)}(\phi, \theta)\right) W^{(s)}(\theta, \phi) \\ \cdot \end{array}\right. \tag{10.75}$$

where $l_{\pm,z}$ are the first-order differential operators

$$l_\pm = \mathrm{e}^{\pm i\phi}\left(\pm\partial_\theta + i\cot\theta\partial_\phi\right), \quad l_z = -i\partial_\phi \tag{10.76}$$

It is worth noting that $l_{\pm,z}$ satisfy the $su(2)$ algebra commutation relations. The $\Lambda_{\pm,0}^{(s)}$ operators have a simple form for the Q and P functions,

$$\begin{aligned} \Lambda_0^{(s)} &= \frac{1}{2}\left(\frac{1}{\varepsilon}\cos\theta + s\cos\theta + s\sin\theta\partial_\theta\right) \\ \Lambda_\pm^{(s)} &= \mathrm{e}^{\pm i\phi}\frac{\sin\theta}{2\varepsilon} \mp \frac{s}{2}\left[\cos\theta l_\pm - \mathrm{e}^{\pm i\phi}\sin\theta\left(l_z \pm 1\right)\right] \end{aligned} \tag{10.77}$$

where $s = \pm 1$ and $\varepsilon = (2S+1)^{-1}$, but for the Wigner function ($s = 0$) they are quite involved

$$\begin{aligned} \Lambda_0^{(0)} &= \frac{1}{4\varepsilon}\cos\theta\Phi(\mathcal{L}^2) - \frac{\varepsilon}{4}\left(\cos\theta + 2\sin\theta\partial_\theta\right)\Phi^{-1}(\mathcal{L}^2) \\ \Lambda_\pm^{(0)} &= \mathrm{e}^{\pm i\phi}\frac{\sin\theta}{4\varepsilon}\Phi(\mathcal{L}^2) \pm \frac{\varepsilon}{4}\left[2\cos\theta l_\pm - \mathrm{e}^{\pm i\phi}\sin\theta\left(2l_z \pm 1\right)\right]\Phi^{-1}(\mathcal{L}^2) \end{aligned} \tag{10.78}$$

where the function $\Phi(\mathcal{L}^2)$ is defined as follows:

$$\Phi(\mathcal{L}^2) = \left[2 - \varepsilon^2\left(2\mathcal{L}^2 + 1\right) + 2\sqrt{\left(1 - \varepsilon^2\mathcal{L}^2\right)^2 - \varepsilon^2}\right]^{1/2} \tag{10.79}$$

and $\mathcal{L}^2$ is the differential realization of the $su(2)$ Casimir operator (10.67).

10.2.6
Evolution Equations

The explicit expression for the star product and/or the correspondence rules (Equations 10.74 to 10.75) allow us to write the evolution equation for the quasi-distribution functions. In the simplest case of *linear* Hamiltonians

$$H = \omega_0 S_z + g S_+ + g^* S_-$$

the evolution has the classical Liouville form

$$\partial_t W^{(s)} = \frac{1}{\sqrt{S\,(S+1)}}\{W^{(s)}, W_H^{(s)}\}_P$$

where the symbol of the Hamiltonian is

$$W_H^{(s)}(\theta,\phi) = \left(\frac{S}{S+1}\right)^{-s/2}\sqrt{S\,(S+1)}\left(\omega_0\cos\theta + g\sin\theta\,\mathrm{e}^{\mathrm{i}\phi} + g^*\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi}\right)$$

and $\{\ldots,\ldots\}_P$ means the Poisson brackets defined on the sphere:

$$\{\ldots,\ldots\}_P = \frac{1}{\sin\theta}\left(\partial_\phi\otimes\partial_\theta - \partial_\theta\otimes\partial_\phi\right)$$

For *nonlinear* Hamiltonians the evolution equations are essentially more involved. In particular, for the simplest nonlinear Hamiltonian, which admits an exact solution, the finite Kerr Hamiltonian

$$H = \chi S_z^2 \tag{10.80}$$

we obtain, taking into account Equation 10.73, the following equations for P, Q, and W functions:

$$\partial_t P = -\sqrt{\frac{(2S+3)(S+1)}{S(2S-1)}}\,[2\,(S+1)\cos\theta + \sin\theta\partial_\theta]\,\partial_\phi P$$

$$\partial_t Q = -\sqrt{\frac{S(2S-1)}{(2S+3)(S+1)}}\,[2S\cos\theta - \sin\theta\partial_\theta]\,\partial_\phi Q$$

$$\partial_t W = -\left[\frac{2S+1}{2}\cos\theta\Phi(\mathcal{L}^2) - \frac{1}{2(2S+1)}\left(\cos\theta + 2\sin\theta\partial_\theta\right)\Phi^{-1}(\mathcal{L}^2)\right]\partial_\phi W$$

10.2.7
Large Representation Dimensions (Semiclassical Limit)

Asymptotic Form of Symbols

Usually, explicit expressions for the symbols of operators and states are quite involved, especially for the Wigner symbols. Nevertheless, in the limit of large spin, $S \gg 1$, such expressions are essentially simplified and allow us to establish an intuitive relation between the operators and their geometric representation.

1. *Angular momentum states* $|k, S\rangle$. For the eigenstates of S_z, we obtain from the integral representations (Equations 10.58 to 10.60)

$$\hat{W}^{(s)}(\mathbf{n}) = \int_0^{2\pi} \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega\,\mathbf{n}\cdot\mathbf{S}}\kappa^{(s)}(\omega)$$

and

$$\begin{aligned} W_k^{(s)}(\theta,\phi) &= \int_0^{2\pi} \mathrm{d}\omega \langle k, S|\,\mathrm{e}^{\mathrm{i}\omega\,\mathbf{n}\cdot\mathbf{S}}|k, S\rangle\kappa^{(s)}(\omega) \\ &= \int_0^{2\pi} \mathrm{d}\omega\, d_{kk}^S(\beta)\,\mathrm{e}^{-\mathrm{i}(\alpha+\gamma)}\kappa^{(s)}(\omega) \end{aligned} \tag{10.81}$$

where we have used the Euler angles to parameterize the group element $\exp(i\omega\, \mathbf{n}\cdot\mathbf{S})$. Recall that the Euler angles α, β, γ are related to the polar parameters (ω, θ, ϕ) according to

$$\sin\frac{\beta}{2} = \sin\theta\sin\frac{\omega}{2}, \quad \tan\frac{\alpha+\gamma}{2} = \cos\theta\tan\frac{\omega}{2}$$

In Equation 10.81,

$$d^S_{kk}(\beta) = \langle k, S|\, \mathrm{e}^{-\mathrm{i}\beta S_y}|k, S\rangle$$

is the Wigner d function. As expected, the symbol (Equation 10.81) does not depend on the angle ϕ, i.e. geometrically it is invariant under rotation around the z axis.

2. $SU(2)$ *coherent states* $|\xi\rangle$. For simplicity, we consider a coherent state located on the equator $|\xi = 1\rangle \equiv |\theta = \pi/2, \phi = 0\rangle$ (which is an eigenstate of the operator S_x, $S_x|\pi/2, 0\rangle = (S/2)|\pi/2, 0\rangle$), such that

$$|\pi/2, 0\rangle = \frac{1}{2^S}\sum_{k=-S}^{S}\sqrt{\frac{(2S)!}{(S+k)!(S-k)!}}|k, S\rangle \tag{10.82}$$

The density matrix is

$$\rho = \sum_{k,n=-S}^{S} c_k c_n^*|k, S\rangle\langle n, S|, \quad c_k = \frac{1}{2^S}\sqrt{\frac{(2S)!}{(S-k)!(S+k)!}}$$

and the integral representation for its symbol takes the form

$$\begin{aligned} W^{(s)}_{\xi=1}(\theta, \phi) &= \int_0^{2\pi} \mathrm{d}\omega\langle\pi/2, 0|\, \mathrm{e}^{\mathrm{i}\omega\,\mathbf{n}\cdot\mathbf{S}}|\pi/2, 0\rangle\kappa^{(s)}(\omega) \\ &= \int_0^{2\pi} \mathrm{d}\omega I(\theta, \phi, \omega)\kappa^{(s)}(\omega) \end{aligned} \tag{10.83}$$

where

$$\begin{aligned} I(\theta, \phi, \omega) &= \sum_{k,n=-S}^{S} c_k c_n^*\langle n, S|\, \mathrm{e}^{-\mathrm{i}\beta S_y}|k, S\rangle\, \mathrm{e}^{-\mathrm{i}\alpha n - i\gamma k} \\ &= \left(\cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2} + i\sin\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2}\right)^{2S} \end{aligned}$$

and (α, β, γ) are the Euler angles, which can be expressed in terms of polar angles according to

$$\cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2} = \cos\frac{\omega}{2}, \quad \tan\frac{\alpha+\gamma}{2} = \cos\theta\tan\frac{\omega}{2}, \quad \frac{\alpha-\gamma}{2} = \phi - \frac{\pi}{2}$$

giving

$$I(\theta, \phi, \omega) = \left(\cos\frac{\omega}{2} - i\sin\frac{\omega}{2}\sin\theta\cos\phi\right)^{2S}$$

The simplest form for $W_k^{(s)}(\theta, \phi)$ and $W_{\xi=1}^{(s)}(\theta, \phi)$ is obtained for the Wigner symbol, $s = 0$. Using the asymptotic form (10.57) of the function $\kappa^{(0)}(\omega)$, we immediately obtain

$$W_k^{(0)}(\theta, \phi) \simeq (-1)^{S+k}\, \mathrm{d}_{kk}^S(2\theta)\left[1 + \frac{k}{S}\cos\theta\right], \quad |k| \sim S \tag{10.84}$$

$$W_{\xi=1}^{(0)}(\theta, \phi) \simeq (\sin\theta\cos\phi)^{2S-1}\left[1 + \sin\theta\cos\phi\right] \tag{10.85}$$

In particular, the Wigner function corresponding to the state $|S, S\rangle$ takes the following simple form:

$$W_{k=S}^{(0)}(\theta, \phi) \simeq \cos^{2S}\theta\,[1 + \cos\theta]$$

It is worth noting that the above equation can be rewritten as follows:

$$W_\rho^{(0)}(\mathbf{n}\,(\theta, \phi)) \simeq f^{2S-1}(\mathbf{n})\left[1 + f(\mathbf{n})\right]$$

where $f(\mathbf{n}) = n_x$. Then, from the covariance property, one can easily recover the Wigner function for an arbitrary coherent state

$$W_{g\circ\rho}^{(0)}(\mathbf{n}\,(\theta, \phi)) = W_\rho(g\circ\mathbf{n}\,(\theta, \phi))$$

Here $g\circ\rho = T(g)\,\varrho T^{-1}(g)$, and $T(g)$ is the operator of finite rotation in the $(2S + 1)$-dimensional representation of the $su(2)$ algebra.

Semiclassical Evolution

The approximate form of the evolution equation in the limit $S \gg 1$, follows from the star-product expression (Equation 10.65),

$$\partial_t W^{(s)} = 2\varepsilon\{W^{(s)}, W_H^{(s)}\}_P + \varepsilon^2 s\left[\mathcal{L}^2\{W^{(s)}, W_H^{(s)}\}_P - \{\mathcal{L}^2 W^{(s)}, W_H^{(s)}\}_P\right] + O(\varepsilon^3)$$

where $\varepsilon = (2S + 1)^{-1} \ll 1$. For the Wigner function, $s = 0$ and the second term disappears:

$$\partial_t W^{(0)} = 2\varepsilon\{W^{(0)}, W_H^{(0)}\}_P + O(\varepsilon^3) \tag{10.86}$$

In other words, the dynamics of $W^{(0)}(\theta, \phi)$ is governed by the classical Liouville equation up to third order on the small parameter ε, which describes the motion of every point of the initial distribution on the sphere along its classical trajectory.

It is worth noting that in the semiclassical limit, the symbol of an operator $\hat{f}$ is scaled by a power of ε: $W_f^{(s)}(\theta, \phi) \sim \varepsilon^{-\deg f}$, where $\deg f$ is the degree of nonlinearity of $\hat{f}$. This allows us to estimate the order of corrections neglected in Equation 10.86. For instance, for quadratic Hamiltonians the correction would be of order ε, while the Liouville term is $\sim\varepsilon^{-1}$.

We would like to stress that in this approach we start formally with the exact evolution equation for the Wigner function (as it follows from the star product (10.65)). Expanding this in powers of a natural small parameter (the inverse representation dimension), we come to a Liuoville type equation (Equation 10.86), which has a typical solution where every point of the initial distribution moves along its classical trajectory. Further on, we can, in principle, find the classical oscillation period as a function of the initial energy, $T(E)$, which may give us information about the quantum transition frequencies, $2\pi/T(E) = \omega(E) \to (E_{k+1} - E_k)/\hbar$ and the number of quantum states in the unit interval of energy.

Example 10.1 As a nontrivial example, we analyze the finite Kerr Hamiltonian (10.80), [118, 119]. The approximate evolution equation for $W(\Omega) \equiv W^{(0)}(\theta, \phi)$ is immediately obtained using Equation 10.73:

$$\partial_t W(\Omega) = -\frac{\chi}{\varepsilon} \cos\theta \partial_\phi W(\Omega) + O(\varepsilon) \tag{10.87}$$

leading to a simple solution

$$W(\Omega|t) = W\big(\theta(t), \phi(t)\big) = W\left(\theta, \phi - \frac{\chi t}{\varepsilon} \cos\theta \middle| t = 0 \right) \tag{10.88}$$

The average values of the spin operators evolve according to Equations 10.16 and 10.19, and, for the initial coherent state Equation 10.82, they can be easily evaluated,

$$\begin{aligned} \langle S_x(t) \rangle &= \sqrt{S(S+1)}\Gamma(S+3/2) \left(\frac{2}{z}\right)^{S+1/2} J_{S+1/2}(z) \\ \langle S_y(t) \rangle &= \langle S_z(t) \rangle = 0, \qquad z = \chi t (2S+1) \end{aligned} \tag{10.89}$$

which is numerically very close to the exact solution (1.30): $\langle S_x(t) \rangle = S \cos^{2S-1} \chi t$

It is more interesting to evaluate spin fluctuations (spin squeezing) using the semiclassical result (10.88). Although analytical integration can be performed and corresponding analytical expressions for $(\Delta s(t))^2$ can be found, we are interested here in the numerical comparison of the the exact results with those obtained from the Wigner function method. We present this comparison in Figure 10.1. One may observe a good agreement between these two curves up to times $\chi t \sim 1$.

Following the strategy outlined in Section 10.2.7, we obtain the semiclassical evolution equation for average values of the spin operators and the second order correlators:

$$\partial_t \langle S_j \rangle = -\mathrm{i}\alpha_S \chi \langle [S_j, S_z^2] \rangle, \quad \partial_t \langle S_{jk} \rangle = -\mathrm{i}\beta_S \chi \langle [S_{jk}, S_z^2] \rangle \tag{10.90}$$

where $S_{jk} = \{S_j, S_k\}_+, j, k = x, y, z$ and

$$\alpha_S = (1 - 4\varepsilon^2)^{-1/2} = 1 + O(\varepsilon^2), \qquad \beta_S = \sqrt{\frac{1 - 4\varepsilon^2}{1 - 9\varepsilon^2}} = 1 + O(\varepsilon^2)$$

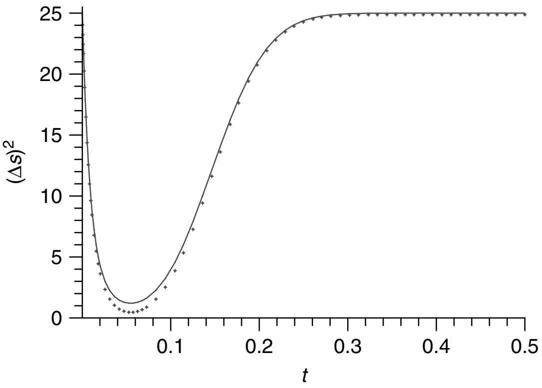

Figure 10.1 Solid line: exact quantum evolution of spin fluctuations $(\Delta s(t))^2$ (Equation 1.31); dotted line: semiclassical approximation for spin fluctuations; $\chi = 1$, $S = 50$.

Clearly, Equation 10.90 coincide with the corresponding averaged quantum Heisenberg equations up to ε^2.

On the other hand, the solution of the classical equations of motions, corresponding to the approximation of decoupled correlators

$$\partial_t\langle S_z\rangle = 0, \quad \partial_t\langle S_x\rangle = -2\chi\langle S_y\rangle\langle S_z\rangle, \quad \partial_t\langle S_y\rangle = 2\chi\langle S_x\rangle\langle S_z\rangle$$

has the form

$$\begin{aligned}\langle S_x(t)\rangle_{\mathrm{cl}} &= \langle S_x(0)\rangle\cos 2\langle S_z(0)\rangle\chi t - \langle S_y(0)\rangle\sin 2\langle S_z(0)\rangle\chi t \\ \langle S_y(t)\rangle_{\mathrm{cl}} &= \langle S_x(0)\rangle\sin 2\langle S_z(0)\rangle\chi t + \langle S_y(0)\rangle\cos 2\langle S_z(0)\rangle\chi t\end{aligned} \tag{10.91}$$

One can see that the "classical solution" diverges from the exact one (Equation 1.31) even for very small times. Indeed, the Taylor expansion of the exact solution (1.31) gives $\langle S_x(t)\rangle \approx S - S(S-1/2)(\chi t)^2 + O\left(t^4\right)$, while the "classical solution" (10.91) has quite a different behavior – for the initial coherent state (10.82) it just maintains a constant value, $\langle S_x(t)\rangle_{\mathrm{cl}} = S$.

One can observe that, according to the solutions (10.88), different parts of the initial distribution rotate with different velocities (depending on the angle θ), which leads to a deformation of the initial distribution (without the self-interference) and eventually to spin squeezing.

Note, that the dynamics of an initial coherent state located close to the pole, where the effect of self-interference becomes important much earlier than for the equatorial coherent states, cannot be well described in terms of classical trajectories (10.88).

In a similar way, the semiclassical approach can be applied to multipartite spin (distinguishable atomic systems). For instance, the Wigner function for bipartite system of two spins S_1 and S_2 is defined as

$$W(\Omega_1, \Omega_2) = \mathrm{Tr}\left(\rho \hat{w}_1^{(0)}(\Omega_1)\hat{w}_2^{(0)}(\Omega_2)\right) \tag{10.92}$$

where $\hat{w}^{(0)}(\Omega)$ is the single particle operator (10.52).

The inversion formula (Equation 10.35) now reads as

$$\rho = \frac{1}{(4\pi)^2 \varepsilon_1 \varepsilon_2} \int d\Omega_1\, d\Omega_2 \hat{w}_1^{(0)}(\Omega_1)\hat{w}_2^{(0)}(\Omega_2) W(\Omega_1, \Omega_2) \tag{10.93}$$

where $\varepsilon_{1,2} = (2S_{1,2} + 1)^{-1}$.

Example 10.2 Let us consider the simplest Hamiltonian that induces quantum correlations for some initial states:

$$H = \chi S_z^{(1)} \otimes S_z^{(2)} \tag{10.94}$$

The equation of motion for the bipartite Wigner function $W(\Omega_1, \Omega_2)$ in the limit of large spins, $S_1, S_2 \gg 1$, can be obtained using the correspondence rules (Equation 10.74)

$$\partial_t W(\Omega_1, \Omega_2) = -\chi \left(\frac{1}{2\varepsilon_2} \cos\theta_2 \partial_{\phi_1} + \frac{1}{2\varepsilon_1} \cos\theta_1 \partial_{\phi_2} \right) W(\Omega_1, \Omega_2) \tag{10.95}$$

with a simple solution

$$W(\Omega_1, \Omega_2|t) = W\left(\theta_1,\ \phi_1 - \frac{\chi t}{2\varepsilon_2}\cos\theta_2;\ \theta_2, ;\ \phi_2 - \frac{\chi t}{2\varepsilon_1}\cos\theta_1 \middle|\ t = 0 \right) \tag{10.96}$$

where the neglected terms are of order $\varepsilon_{1,2}$. One can observe that the Wigner function (10.96) describes a sort of ''entanglement'' in the sense that the dynamic variables of both systems become interdependent during evolution. For initial pure states $|\psi(0)\rangle$, we quantify entanglement in terms of partial trace of the density matrix, $\boldsymbol{\rho}_1(t) = \mathrm{Tr}_2|\psi(t)\rangle\langle\psi(t)|$, so that the entanglement measure we adopt is

$$P(t) = \mathrm{Tr}_1 \rho_1^2(t) \tag{10.97}$$

In terms of the Wigner function, this reads as

$$P(t) = \frac{1}{4\pi\varepsilon_1}\left(\frac{1}{4\pi\varepsilon_2}\right)^2 \int d\Omega_1\, d\Omega_2\, d\Omega_2' W(\Omega_1, \Omega_2|t) W(\Omega_1, \Omega_2'|t) \tag{10.98}$$

Let as assume that initially each subsystem is in the equatorial coherent state (10.82):

$$\begin{aligned} |\psi(0)\rangle &= \left|\theta_1 = \frac{\pi}{2},\ \varphi_1 = 0\right\rangle_1 \left|\theta_2 = \frac{\pi}{2},\ \varphi_2 = 0\right\rangle_2 \\ &= \frac{1}{2^{S_1+S_2}} \sum_{k=-S_1}^{S_1} \sum_{n=-S_2}^{S_2} \gamma_{S_1 k}\gamma_{S_2 n} |k, S_1\rangle_1 |n, S_2\rangle_2 \end{aligned} \tag{10.99}$$

where γ_{Sk} are defined in Equation 10.82. The above state is not an eigenstate of the Hamiltonian (10.94), and therefore, spin subsystems get entangled in the course of evolution. This entanglement has a periodic nature and can be easily characterized analytically owing to the simplicity of Equation 10.94. In particular, the reduced density matrix is

$$\rho_1(t) = \frac{1}{2^{2(S_1+S_2)}} \sum_{k_1,k_2=-S_1}^{S_1} \gamma_{S_1k_1}\gamma_{S_1k_2}\, \mathrm{e}^{\mathrm{i}\chi t(k_1-k_2)S_2} \times \left(1+\mathrm{e}^{-\mathrm{i}\chi t(k_1-k_2)}\right)^{2S_2} |k_1, S_1\rangle_{11}\langle k_2, S_1|$$

and the exact expression for the entanglement measure (Equation 10.97) takes the form

$$P(t) = \frac{1}{2^{4S_2}} \sum_{n=0}^{4S_2} C_n^{4S_2} \cos^{4S_1} \frac{\chi t(n-2S_2)}{2} \tag{10.100}$$

where $C_n^{4S_2} = \frac{n!}{(4S_2)!(n-4S_2)!}$ are binomial coefficients.

On the other hand, in the limit of large spins, we can estimate the entanglement using Equation 10.98, taking into account the approximate form of the bipartite Wigner function

$$\begin{aligned} W(\Omega_1, \Omega_2|t) \simeq & \left(\sin\theta_1 \cos(\phi_1 - \frac{\chi t}{2\varepsilon_2}\cos\theta_2)\right)^{2S_1-1} \\ & \times \left(1+\sin\theta_1 \cos(\phi_1 - \frac{\chi t}{2\varepsilon_2}\cos\theta_2)\right) \\ & \times \left(\sin\theta_2 \cos(\phi_2 - \frac{\chi t}{2\varepsilon_1}\cos\theta_1)\right)^{2S_2-1} \\ & \times \left(1+\sin\theta_2 \cos(\phi_2 - \frac{\chi t}{2\varepsilon_1}\cos\theta_1)\right) \end{aligned} \tag{10.101}$$

where the approximate expression (10.85) for the coherent state of the type (10.82) has been used.

In Figure 10.2 the evolution of the entanglement measure obtained from the semiclassical approach (Equations 10.100 and 10.98) is compared to the exact result (Equation 10.101). Two spins are taken to be $S_1 = S_2 = 12$. One can observe a remarkable coincidence of these curves.

10.3
Field Quasi-distributions

10.3.1
P Function

According to the general approach outlined in Section 10.2 of this chapter, the $P(\alpha)$ function is introduced as the coefficient of expansion of the field density matrix ρ over the coherent states:

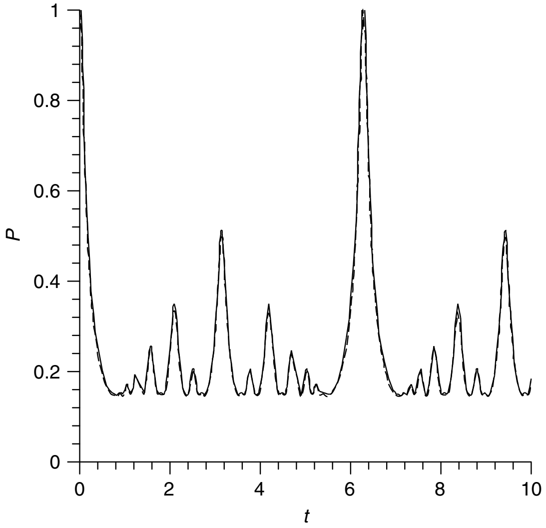

Figure 10.2 Evolution of the entanglement measure (10.97) from exact expression (10.100) (solid line); from semiclassical approximation (dotted line); $S_1 = S_2 = 12$.

$$\rho = \frac{1}{\pi} \int d^2\alpha \, P(\alpha) \, |\alpha\rangle\langle\alpha| \tag{10.102}$$

The normalization condition

$$\mathrm{Tr}\,\rho = \frac{1}{\pi} \int d^2\alpha \, P(\alpha) = 1$$

follows from the unitary trace $\mathrm{Tr}\,\rho = 1$. To invert Equation 10.102 we multiply it by $\exp(\gamma(a^\dagger - \beta^*)) \exp\left(-\gamma^*(a - \beta)\right)$, so that after tracing and integrating over γ one obtains

$$P(\alpha) = \frac{1}{\pi} \int d^2\gamma \, e^{\gamma^*\beta - \gamma\beta^*} \, Tr\left(\rho \, e^{\gamma a^\dagger} \, e^{-\gamma^* a}\right) = \mathrm{Tr}\left(\rho \hat{w}_P(\alpha)\right)$$

where

$$\hat{w}_P(\alpha) = \frac{1}{\pi} \int d^2\gamma \, e^{\gamma(a^\dagger - \beta^*)} e^{-\gamma^*(a - \beta)}$$

Usually, the P function is singular and may contain the Dirac δ function and its derivatives. For example, for a coherent state $|\beta\rangle$,

$$\rho(\beta) = |\beta\rangle\langle\beta| = \int d^2\alpha \, \delta^{(2)}(\alpha - \beta)|\alpha\rangle\langle\alpha|$$

which means that the function $\mathcal{P}$ of the coherent state $|\beta\rangle$ is

$$P_\beta(\alpha) = \pi\delta^{(2)}(\alpha - \beta)$$

The P-symbol for any operator $\hat{f}$ is defined in the same way as in Equation 10.102

$$\hat{f} = \frac{1}{\pi} \int d^2\alpha \, P_f(\alpha) \, |\alpha\rangle\langle\alpha|$$

10.3.2
Q Function

The Q function is defined as the average value of the density matrix in the coherent state

$$Q(\alpha) = \langle\alpha|\rho|\alpha\rangle = \mathrm{Tr}\,(\rho|\alpha\rangle\langle\alpha|) = \mathrm{Tr}\left(\rho\hat{w}_Q(\alpha)\right) \qquad (10.103)$$

where (see Equation 3.77)

$$\hat{w}_Q(\alpha) = \frac{1}{\pi}\int \mathrm{d}^2\gamma\, \mathrm{e}^{\gamma(a^\dagger-\beta^*)}\mathrm{e}^{-\gamma^*(a-\beta)-|\gamma|^2}$$

The Q function is obviously normalized according to

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha\, Q(\alpha) = 1$$

To invert Equation 10.103 we make use of the expansion of the density matrix in the basis of the displacement operators (3.75), so that after multiplying (Equation 10.103) by $\exp(\alpha\beta^* - \alpha^*\beta)$ and integrating over α, we obtain

$$Tr\left(\rho D(\beta)\right) = \frac{1}{\pi^2}\,\mathrm{e}^{|\beta|^2}\int \mathrm{d}^2\alpha Q(\alpha)\,\mathrm{e}^{\alpha\beta^*-\alpha^*\beta}$$

and finally,

$$\rho = \frac{1}{\pi^2}\int d^2\beta\,\mathrm{e}^{-\beta a^\dagger}\,\mathrm{e}^{-\beta^* a}\int \mathrm{d}^2\alpha Q(\alpha)\,\mathrm{e}^{\alpha\beta^*-\alpha^*\beta}$$

It is easy to relate the functions $P(\alpha)$ and $Q(\alpha)$:

$$Q(\alpha) = \langle\alpha|\rho|\alpha\rangle = \int \mathrm{d}^2\beta\, P(\beta)\,\langle\beta|\alpha\rangle\langle\alpha|\beta\rangle = \int \mathrm{d}^2\beta\, P(\beta)\,\mathrm{e}^{-|\alpha-\beta|^2} \qquad (10.104)$$

One can see that the $Q(\alpha)$ function is in a way a smoothed $P(\alpha)$ function. Because of such "smoothness", it is quite appropriate to use the Q function for representation purposes. For instance, the Q function of the coherent state $|\beta\rangle$ is $Q_\beta(\alpha) = |\langle\alpha|\beta\rangle|^2 = \mathrm{e}^{-|\alpha-\beta|^2}$, while for the number state $|n\rangle$ it has the form $Q_n(\alpha) = |\langle\alpha|n\rangle|^2 = \mathrm{e}^{-|\alpha|^2}|\alpha|^{2n}/n!$. Usually, the function $Q(\alpha)$ is represented by contour lines in the phase plane ($q = \sqrt{2}\mathrm{Re}\alpha,\ p = \sqrt{2}\mathrm{Im}\alpha$). In this case, the function $Q(\alpha)$ for a coherent state $|\beta\rangle$ is represented by circles centered at the point ($q_0 = \sqrt{2}\mathrm{Re}\,\beta,\ p_0 = \sqrt{2}\mathrm{Im}\beta$) and mainly localized within a circle of unit radius. A number state $|n\rangle$ is represented as a ring centered at the origin of the phase plane with the radius $\sqrt{n}$ and unit width.

Defining the Q-symbol of an operator $\hat{f}$ as $Q_f(\alpha) = \langle\alpha|\hat{f}|\alpha\rangle$, we can represent its expectation value $\langle\hat{f}\rangle = Tr\left(\hat{f}\rho\right)$ using Equation 10.102 as follows

$$\langle\hat{f}\rangle = \mathrm{Tr}\left(\int \frac{\mathrm{d}^2\alpha}{\pi}\,P(\alpha)|\alpha\rangle\langle\alpha|\hat{f}\right) = \int \frac{\mathrm{d}^2\alpha}{\pi}P(\alpha)Q_f(\alpha) = \int \frac{\mathrm{d}^2\alpha}{\pi}\,P_f(\alpha)\,Q(\alpha)$$

10.3.3
Wigner Function

The Wigner operator is introduced as

$$\hat{w}(\alpha) = \frac{1}{\pi}\int d^2\eta\, e^{\eta^*\alpha-\alpha^*\eta}\, D(\eta) \tag{10.105}$$

where $D(\eta)$ is the displacement operator (3.55) and the Wigner function of the density matrix ρ is

$$W(\alpha) = \mathrm{Tr}\left(\hat{w}(\alpha)\rho\right) = \frac{1}{\pi}\int d^2\eta\, e^{\eta^*\alpha-\alpha^*\eta}\,\mathrm{Tr}\left(D(\eta)\rho\right) \tag{10.106}$$

We also define the Wigner symbol for any operator $\hat{f}$ as

$$W_f(\alpha) = \mathrm{Tr}\left(\hat{w}(\alpha)\hat{f}\right) \tag{10.107}$$

The inversion operation for Equation 10.106 is very simple

$$\rho = \frac{1}{\pi}\int d^2\alpha\hat{w}(\alpha)\, W(\alpha)$$

The Wigner function is normalized according to

$$\frac{1}{\pi}\int d^2\alpha\, W(\alpha) = \mathrm{Tr}\rho = 1$$

The self-duality property allows us to calculate the expectation value of an operator $\hat{f}$ in a state with the density matrix ρ as

$$\langle \hat{f}\rangle = \mathrm{Tr}\left(\hat{f}\rho\right) = \frac{1}{\pi}\int d^2\alpha\, W(\alpha)\, W_f(\alpha)$$

In particular, if $\hat{f} = \rho$,

$$\frac{1}{\pi}\int d^2\alpha\, W^2(\alpha) = \mathrm{Tr}\rho^2$$

We note that after the change of variables

$$a = \frac{\hat{q}+i\hat{p}}{\sqrt{2}}, \quad \alpha = \frac{q+ip}{\sqrt{2}}, \quad \eta = \frac{-\gamma_1+i\gamma_2}{\sqrt{2}}, \quad D(\alpha) \equiv e^{i(\gamma_1\hat{p}+\gamma_2\hat{q})}$$

we can represent the Wigner operator in a different form:

$$\hat{w}(\alpha) \equiv \hat{w}(p,q) = \frac{1}{2\pi}\int d\gamma_1 d\gamma_2\, e^{i\gamma_1(\hat{p}-p)+i\gamma_2(\hat{q}-q)}$$

It is clear that in the classical limit, when the operators $\hat{p}$ and $\hat{q}$ commute, the Wigner operator is reduced to

$$\hat{w}(p,q) \to 2\pi\delta(\hat{p}-p)\delta(\hat{q}-q)$$

The Wigner function $W(p,q)$ is a nonsingular function, but it can take on negative values. It can be proven that if the Wigner function of a pure state is positively defined, then, this function is a Gaussian.

The connection with functions Q and P is given by the equations

$$W(\alpha) = \frac{2}{\pi}\int P(\beta)\,\mathrm{e}^{-2|\alpha-\beta|^2}\,d^2\beta\,, \quad Q(\alpha) = \frac{2}{\pi}\int W(\beta)\,\mathrm{e}^{-2|\alpha-\beta|^2}\,d^2\beta$$

For a coherent state $|\beta\rangle$, we have

$$W_\beta(\alpha) = 2\,\mathrm{e}^{-2|\alpha-\beta|^2}$$

All these distributions are related to the Heisenberg–Weyl group and can be considered to be particular cases of a more general construction – the s-ordered distribution function, which we consider next.

10.3.4
s-Ordered Distributions

Let us introduce the following operator [120]

$$\hat{w}^{(s)}(\alpha) = \frac{1}{\pi}\int \mathrm{d}^2\xi\, D(\xi)\,\mathrm{e}^{\alpha\xi^* - \alpha^*\xi\ + s|\xi|^2/2} \tag{10.108}$$

where $D(\xi)$ is the displacement operator (3.55). Note that the operator (10.108) can be represented by

$$\hat{w}^{(s)}(\alpha) = D(\alpha)T(s)D^\dagger(\alpha) \tag{10.109}$$

where

$$T(s) = \frac{1}{\pi}\int \mathrm{d}^2\xi\, D(\xi)\,\mathrm{e}^{s|\xi|^2/2} = \frac{2}{1-s}\left(\frac{s+1}{s-1}\right)^{a^\dagger a} \tag{10.110}$$

$$\mathrm{Tr}T(s) = 1, \quad T^\dagger(s) = T(s)$$

The above integral converges for values $s < 1$. In particular,

$$T(0) = 2(-1)^{a^\dagger a}, \quad T(-1) = |0\rangle\langle 0| \tag{10.111}$$

$$T(s)|\alpha\rangle = \frac{2}{1-s}\exp\left[\frac{|\alpha|^2}{2}\left(\left|\frac{s+1}{s-1}\right|^2 - 1\right)\right]|\frac{s+1}{s-1}\alpha\rangle$$

Note that the operator $T(0)$ is proportionate to the parity operator, so we have

$$T(0)f(a,a^\dagger)T^{-1}(0) = f(-a,-a^\dagger)$$

where $f(a,a^\dagger)$ is an arbitrary function of the creation–annihilation operators.

It can easily be seen that the operator $\hat{w}(\alpha,s)$ is Hermitian and normalized

$$\hat{w}^{(s)}(\alpha) = \hat{w}^{(s)}(\alpha)^\dagger, \quad \mathrm{Tr}\,\hat{w}^{(s)}(\alpha) = 1, \quad \frac{1}{\pi}\int \mathrm{d}^2\alpha\,\hat{w}^{(s)}(\alpha) = I \tag{10.112}$$

and also satisfies the following overlap relations:

$$\mathrm{Tr}\,\left[\hat{w}^{(s)}(\alpha)\hat{w}^{(-s)}(\beta)\right] = \pi\delta^2(\alpha-\beta) \tag{10.113}$$

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha\,\left(\hat{w}^{(s)}(\alpha)\right)_{mm'}\left(\hat{w}^{(-s)}(\alpha)\right)_{kk'} = \delta_{mk'}\delta_{m'k} \tag{10.114}$$

We can observe that the operators $\hat{w}^{(s)}(\alpha)$ and $\hat{w}^{(s')}(\alpha)$, $s' > s$ are related to each other by the following relation:

$$\begin{aligned}\hat{w}^{(s')}(\alpha) &= \frac{2}{\pi\,(s'-s)}\int \mathrm{d}^2\beta\hat{w}^{(s)}(\beta)\exp\left[-\frac{2}{(s'-s)}|\alpha-\beta|^2\right]\\ &= \exp\left[\frac{s'-s}{2}\frac{\partial^2}{\partial\alpha\partial\beta}\right]\hat{w}^{(s)}(\alpha)\end{aligned}$$

We assign an s-ordered symbol to any operator $\hat{f}$ according to

$$W_f^{(s)}(\alpha) = \mathrm{Tr}\left(\hat{f}\hat{w}^{(s)}(\alpha)\right)$$

In a particular case of the density matrix, the corresponding symbol is called the s-ordered distribution function:

$$W^{(s)}(\alpha) = \mathrm{Tr}\left(\hat{w}^{(s)}(\alpha)\rho\right)$$

This distribution function has the following properties, which are the consequence of Equations 10.112 to 10.114:

1. $W^{(s)}(\alpha)$ is real:

$$W^{(s)}(\alpha) = W^{(s)}(\alpha)^* \tag{10.115}$$

2. $W^{(s)}(\alpha)$ is normalized:

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha\, W^{(s)}(\alpha) = 1 \tag{10.116}$$

3. In terms of the density matrix, the function $W^{(s)}(\alpha)$ is given as:

$$W^{(s)}(\alpha) = \frac{2}{1-s}\sum_{n=0}^{\infty}\left(\frac{s+1}{s-1}\right)^n \langle n|D^{\dagger}(\alpha)\rho D(\alpha)|n\rangle \tag{10.117}$$

4. The following overlap relation is satisfied:

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha\, W_f^{(s)}(\alpha)\, W_g^{(-s)}(\alpha) = \mathrm{Tr}(\hat{f}\hat{g}) \tag{10.118}$$

5. The following inversion formula is valid:

$$\rho = \frac{1}{\pi}\int \mathrm{d}^2\alpha\, \hat{w}^{(-s)}(\alpha)\, W^{(s)}(\alpha) \tag{10.119}$$

As a simple example, we write the s-symbols of the operators $a, a^\dagger$ and $a^\dagger a$:

$$W_a^{(s)}(\alpha) = \alpha, \quad W_{a^\dagger}^{(s)}(\alpha) = \alpha^*, \quad W_{a^\dagger a}^{(s)}(\alpha) = |\alpha|^2 - \frac{s+1}{2}$$

By direct calculations, it is easy to prove the following relations satisfied by the operator $\hat{w}^{(s)}(\alpha)$:

$$\begin{aligned} a\hat{w}^{(s)}(\alpha) &= \left(\alpha - \frac{1+s}{2}\partial_{\alpha^*}\right)\hat{w}^{(s)}(\alpha), \quad a^\dagger \hat{w}^{(s)}(\alpha) = \left(\alpha^* + \frac{1-s}{2}\partial_\alpha\right)\hat{w}^{(s)}(\alpha) \\ \hat{w}^{(s)}(\alpha)a &= \left(\alpha + \frac{1-s}{2}\partial_{\alpha^*}\right)\hat{w}^{(s)}(\alpha), \quad \hat{w}^{(s)}(\alpha)a^\dagger = \left(\alpha^* - \frac{1+s}{2}\partial_\alpha\right)\hat{w}^{(s)}(\alpha) \end{aligned} \tag{10.120}$$

From Equation 10.120, it is easy to obtain the following correspondence rules:

$$\begin{aligned} a\rho &\to \left(\alpha + \tfrac{1-s}{2}\partial_{\alpha^*}\right)W^{(s)}(\alpha), \quad a^\dagger\rho \to \left(\alpha^* - \frac{1+s}{2}\partial_\alpha\right)W^{(s)}(\alpha) \\ \rho a &\to \left(\alpha - \tfrac{1+s}{2}\partial_{\alpha^*}\right)W^{(s)}(\alpha), \quad \rho a^\dagger \to \left(\alpha^* + \frac{1-s}{2}\partial_\alpha\right)W^{(s)}(\alpha) \end{aligned} \tag{10.121}$$

In general, the star product is defined as [109]

$$\begin{aligned} W_f^{(s)}(\alpha) * W_g^{(s)}(\alpha) = \exp&\left[-\frac{s}{2}\left(\partial_\alpha^{(f)}\partial_{\alpha^*}^{(g)} + \partial_{\alpha^*}^{(f)}\partial_\alpha^{(g)}\right)\right] \\ &\exp\left[\frac{1}{2}\left(\partial_{\alpha^*}^{(f)}\partial_\alpha^{(g)} - \partial_\alpha^{(f)}\partial_{\alpha^*}^{(g)}\right)\right] W_f^{(s)}(\alpha)W_g^{(s)}(\alpha) \end{aligned} \tag{10.122}$$

Equation 10.119 is fundamental for the so-called quantum tomography, a method that allows to reconstruct the density matrix by measuring the probabilities-diagonal elements of the displaced density matrix. In particular, for the value of the parameter $s = 0$, using Equations 10.117 and 10.119, we obtain

$$\rho = \frac{2}{\pi}\int \mathrm{d}^2\alpha\hat{w}^{(0)}(\alpha)\sum_{n=0}^{\infty}(-1)^n\,\langle n|D^\dagger(\alpha)\rho D(\alpha)|n\rangle$$

where

$$\hat{w}^{(0)}(\alpha) = 2D(\alpha)(-1)^{a^\dagger a}D^\dagger(\alpha)$$

We consider some particular cases:

1. In the case when $s = -1$ it can easily be seen that the function $W^{(-1)}(\alpha)$ is the Husimi $Q(\alpha)$ function (10.104). Since

$$\hat{w}^{(-1)}(\alpha) = D(\alpha)|0\rangle\langle 0|D^\dagger(\alpha) = |\alpha\rangle\langle\alpha|$$

we have

$$W^{(-1)}(\alpha) = \langle 0|D^{\dagger}(\alpha)\rho D(\alpha)|0\rangle = Q(\alpha)$$

2. In the case when $s = 0$, the function $W^{(s=0)}(\alpha)$ coincides with the Wigner function $W(\alpha)$ (10.105), (10.106)

$$\hat{w}^{(0)}(\alpha) = 2D(\alpha)(-1)^{a^{\dagger}a}D^{\dagger}(\alpha)$$

$$W^{(0)}(\alpha) = 2\sum_{n=0}^{\infty}(-1)^{n}\,\langle n|D^{\dagger}(\alpha)\rho D(\alpha)|n\rangle$$

or in the integral representation

$$W^{(0)}(\alpha) = \frac{1}{\pi}\int \mathrm{d}^2\xi\ \mathrm{e}^{\alpha\xi^{*}-\alpha^{*}\xi}\ \mathrm{Tr}(\rho D(\xi))$$

3. In the marginal case when $s = 1$, the integral (10.110) formally diverges. However, using the inversion formula (10.119) and Equation 10.111 we obtain

$$\rho = \frac{1}{\pi}\int \mathrm{d}^2\alpha D(\alpha)|0\rangle\langle 0|D^{\dagger}(\alpha)W^{(1)}(\alpha) = \frac{1}{\pi}\int \mathrm{d}^2\alpha|\alpha\rangle\langle\alpha|W^{(1)}(\alpha)$$

thus the function $W^{(1)}(\alpha)$ is the Glauber function $P(\alpha)$ (10.102): $W^{(1)}(\alpha) = P(\alpha)$.

Constructions similar to the *s*-ordered quasi-distribution functions can also be developed for more complicated dynamic groups.

10.4
Miscellaneous Applications

Here we consider two representative examples that apply the method of the quasi-distribution function. The first system, which we treat in the semiclassical way, is the well-known Kerr Hamiltonian. In this example, we just demonstrate that such an effect as field squeezing can be treated semiclassically. The second example is a nontrivial application of semiclassical ideology to the Dicke model in the most nonlinear regime, where all the others perturbation approaches fail to predict both the spectrum and the system dynamics.

10.4.1
Kerr Hamiltonian

Let us consider the simplest nonlinear Hamiltonian, already analyzed in Chapter 4 (slightly modified to avoid an unnecessary phase factor)

$$H = \frac{\chi}{2}\left(\hat{a}^{\dagger}\right)^2 \hat{a}^2 \tag{10.123}$$

The Wigner function ($s = 0$) obeys the following differential equation:

$$\begin{aligned} i\partial_t W^{(0)} &= \chi\left(|\alpha|^2 - 1\right)\left(\alpha^* \frac{\partial W^{(0)}}{\partial \alpha^*} - \alpha \frac{\partial W^{(0)}}{\partial \alpha}\right) \\ &\quad - \frac{\chi}{4}\left(\alpha^* \frac{\partial^3 W^{(0)}}{\partial \alpha^* \partial \alpha^* \partial \alpha} - \alpha \frac{\partial^3 W^{(0)}}{\partial \alpha \partial \alpha \partial \alpha^*}\right) \end{aligned}$$

which in polar coordinates, $\alpha = r\,\mathrm{e}^{\mathrm{i}\phi}$, reads

$$\partial_t W^{(0)} = \chi(r^2 - 1)\frac{\partial W^{(0)}}{\partial \phi} - \frac{\chi}{16}\left(\frac{1}{r^2}\frac{\partial^3}{\partial \phi^3} + \frac{\partial^3}{\partial r^2 \partial \phi} + \frac{1}{r}\frac{\partial^3}{\partial \phi^2 \partial r}\right) W^{(0)} \tag{10.124}$$

In the semiclassical limit, $|\alpha|^2 = r^2 \gg 1$, the Wigner function dynamic is approximately (for times $\chi t \ll 1$) described by the first order equation

$$\partial_t W^{(0)} = \chi(r^2 - 1)\partial_\phi W^{(0)}$$

which can be easily solved:

$$W^{(0)}(r, \phi|t) = W^{(0)}(r, \phi = \phi_0 - \chi t + r^2 \chi t)$$

In particular, for the initial coherent state $|\beta\rangle$, the Wigner function acquires the form

$$W^{(0)}_{\beta}(r, \phi|t) = 2\exp\left(-2\left|r\,\mathrm{e}^{\mathrm{i}\phi_0 - i\chi t + ir^2\chi t} - \beta\right|^2\right) \tag{10.125}$$

so that the average value of any field operator can be computed using the standard form (10.13), for instance,

$$\langle a(t)\rangle = \frac{1}{\pi}\int \mathrm{d}^2\alpha\, W^{(0)}(\alpha|t)\, W^{(0)}_a(\alpha)$$

where $W^{(0)}_a(\alpha) = \alpha$, giving

$$\langle a(t)\rangle = \frac{\beta}{(1 + i\chi t/2)^2}\exp\left(-2|\beta|^2 + i\chi t + \frac{2|\beta|^2}{1 + i\chi t/2}\right)$$

which coincides with the exact result

$$\left\langle \hat{a}(t)\right\rangle = \beta\,\mathrm{e}^{-|\beta|^2[1 - \exp(i\chi t)]} \tag{10.126}$$

for times $\chi t \ll 1$. In a similar way, the second-order moments are calculated:

$$\langle a^2(t)\rangle = \frac{1}{\pi}\int \mathrm{d}^2\alpha\, W(\alpha)\, W_{a^2}(\alpha) = \frac{1}{\pi}\int \mathrm{d}^2\alpha\, W(\alpha)\alpha^2$$

giving

$$\langle a^2(t)\rangle = \frac{\beta^2}{(1 + i\chi t)^3}\exp\left(-2|\beta|^2 + 2i\chi t + \frac{2|\beta|^2}{1 + i\chi t}\right)$$

Taking into account that $\langle a^\dagger a(t)\rangle = |\beta|^2$, we obtain for the fluctuation of $\hat{q}_\theta = (a\,\mathrm{e}^{\mathrm{i}\theta} + a^\dagger \mathrm{e}^{-\mathrm{i}\theta})/\sqrt{2}$,

$$S = \min_\theta \sigma_{q_\theta} = 1 + 2\left(\left\langle a^\dagger a\right\rangle - |\langle a\rangle|^2\right) - 2\left|\left\langle a^2\right\rangle - \langle a\rangle^2\right|$$

the following expression

$$S = 1 + 2r^2 - r^3\tau + r\tau - 2r\sqrt{1+r^2} + r^2\tau\frac{2+3r^2}{\sqrt{1+r^2}} \tag{10.127}$$

where $r = |\beta|^2\chi t, \tau = \chi t$. In the limit $r \gg 1$, Equation 10.127 immediately leads to

$$S \approx \frac{1}{4r^2} + 2r^3\tau$$

giving the minimum value of S as well as the best squeezing time $\tau_{\min}$

$$S_{\min} \approx 2^{1/3}|\beta|^{-2/3}, \quad \tau_{\min} \approx 4^{-1/3}|\beta|^{-5/3}$$

which coincide with the result obtained from the exact calculations [121] in the same limit $r \gg 1$.

10.4.2
The Dicke Hamiltonian

As we have already seen, for the Dicke model, the interaction Hamiltonian is

$$H_{\text{int}} = g\left[a^\dagger S_- + aS_+\right]$$

(we focus on the resonance case only). In the regime where the number of excitations is less than the number of atoms, $N < A$, it can be rewritten as follows (Equation 6.31):

$$H_{\text{int}} = g\left(L_+\sqrt{A - N/2 + L_z} + \sqrt{A - N/2 + L_z}L_-\right)$$

where $L_{\pm,z}$ are generators of $(N+1)$-dimensional representation of the $su(2)$ algebra. In Chapter 6, we have studied in detail the quasi-linear cases, $N \ll A$ and $A \ll N$, when the square roots can be expanded in series of the small parameter $\epsilon = (A - N/2 + 1/2)^{-1}$, or $\epsilon = (N - A/2 + 1/2)^{-1}$ so that nonlinear terms in the $su(2)$ generators were taken into account in a perturbative manner. The case when $N \sim A$ is essentially nonlinear and the standard perturbation theory fails to describe even short-time behavior in this regime. Here, we consider the most nonlinear case the – Dicke Hamiltonian, when $A = N + 1$. We study the system dynamics in a semiclassical limit $N = 2S \gg 1$. The interaction Hamiltonian takes the form

$$H_{\text{int}} = g\left(S_+\sqrt{S + 1 + S_z} + \sqrt{S + 1 + S_z}S_-\right) \tag{10.128}$$

where we have used notations of angular momentum operators $(S_{\pm,z})$, acting in the $(2S+1)$-dimensional Hilbert space in a standard way.

We start the calculation of the s-ordered symbols (restricting ourselves to $s = -1$ and $s = 0$) of the Hamiltonian (10.128) with the observation that the interaction Hamiltonian can be rewritten as follows:

$$H_{\text{int}} = g\sqrt{\frac{2}{\varepsilon}} \sum_{n=0}^{\infty} c_n (\varepsilon/2)^n \{S_z, \{\ldots\{S_z, S_x\}_+\}\}_+$$

where $\varepsilon = (2S+1)^{-1}$ and the coefficients c_n are defined from the square root expansion

$$\sqrt{1+x} = \sum_{n=0}^{\infty} c_n x^n$$

Making use of the explicit expression for the correspondence rules (Equations 10.74 and 10.75), we may see that

$$W^{(s)}_{\{S_z,S_x\}_+}(\Omega) = W^{(s)}_{S_z}(\Omega) * W^{(s)}_{S_x}(\Omega) + W^{(s)}_{S_x}(\Omega) * W^{(s)}_{G}(\Omega) = 2\Lambda_0^{(s)}(\phi,\theta)\, W^{(s)}_{S_x}(\Omega)$$

where

$$W^{(s)}_{S_x}(\Omega) = \left(\frac{S}{S+1}\right)^{-s/2} \sqrt{S(S+1)}\cos\phi\sin\theta$$

and $\Lambda_0^{(s)}(\phi,\theta)$ is defined in Equations 10.77 and 10.78. Thus, the symbol of Equation 10.128 is obtained using the star-product technology:

$$W^{(s)}_{H_{\text{int}}}(\Omega) = 2g\sqrt{S(S+1)}\left(\frac{S}{S+1}\right)^{-s/2} \sqrt{S + 1/2 + \Lambda_0^{(s)}(\phi,\theta)\cos\phi\sin\theta} \tag{10.129}$$

In the semiclassical limit, $S \gg 1$, the expression for $\Lambda_0^{(0)}(\phi,\theta)$ is simplified

$$\Lambda_0^{(0)}(\phi,\theta) \approx \frac{1}{2\varepsilon}\cos\theta - \frac{\varepsilon}{4}\left(\cos\theta + \sin\theta\partial_\theta + \cos\theta\mathcal{L}^2\right)$$

so that in the leading order of magnitude we reduce Equation 10.129 for $s = 0$ (Wigner function) to

$$W^{(0)}_{H_{\text{int}}}(\Omega) \approx 2g\sqrt{\frac{1-\varepsilon^2}{(2\varepsilon)^3}} \sqrt{1 + \cos\theta\left(1 - \frac{\varepsilon^2}{2}\right)\left(1 + \tan\theta\partial_\theta + \mathcal{L}^2\right)\cos\phi\sin\theta}$$

Taking into account that $\left(\tan\theta\partial_\theta + \mathcal{L}^2\right)\cos\phi\sin\theta = 3\cos\phi\sin\theta$, we finally obtain

$$W^{(0)}_{H_{\text{int}}}(\Omega) \approx 2g\sqrt{\frac{1-3\varepsilon^2}{(2\varepsilon)^3}} \sqrt{1 + 2\varepsilon^2 + \cos\theta\cos\phi\sin\theta} \tag{10.130}$$

The exact Q-symbol of the interaction Hamiltonian is immediately calculated since $\Lambda_0^{(-1)} W^{(-1)}_{S_x}(\Omega) = (S - 1/2)\cos\theta W^{(-1)}_{S_x}(\Omega)$, giving

$$W^{(-1)}_{H_{\text{int}}}(\Omega) \approx 2gS^{3/2}\sqrt{1 + (1-2\varepsilon)\cos\theta\cos\phi\sin\theta} \tag{10.131}$$

which in the leading order on the parameter ε coincides with the naive expansion for the classical Hamiltonian,

$$H_{\text{cl}} = 2gS\sqrt{1 + S + S\cos\theta\cos\phi\sin\theta}$$

obtained from Equation 10.128 by substituting $S_z \to S\cos\theta,\ S_\pm \to S\sin\theta\,\mathrm{e}^{\pm i\phi}$.

As was discussed in Section 10.2.6, in the evolution equation for the Wigner function the first-order corrections to the classical equations of motion disappear in the semiclassical limit (see Equation 10.86), and the description of the quantum evolution in terms of the Wigner function is very convenient in the vicinity of the stable points of the corresponding classical motion. In the region close to the classical separatrix, we cannot, in general, arrive at such a conclusion. In this section, we study the semiclassical dynamics of the Dicke model in the semiclassical limit, using, as a basic tool the Q-symbol of the interaction Hamiltonian (10.131), which is suitably rewritten as follows:

$$W^{(-1)}_{H_{\text{int}}}(\Omega) \approx 2gS^{3/2}\sqrt{1 + 2\varepsilon + \cos\theta\cos\phi\sin\theta} \tag{10.132}$$

The symbol (Equation 10.132) can be considered as the classical energy, and thus, provides the information about stable points, where

$$\partial_\theta W^{(-1)}_{H_{\text{int}}}(\Omega_0) = \partial_\phi W^{(-1)}_{H_{\text{int}}}(\Omega_0) = 0$$

We have the stable points

$$\cos\theta_0 \approx \frac{1}{3} - \frac{1}{6S}, \quad \phi_0 = 0,\ \pi$$

The semiclassical evolution is defined by the classical trajectories on the sphere: $W^{(-1)}_{H_{\text{int}}}(\theta, \phi|t) = W^{(-1)}_{H_{\text{int}}}(\theta_0, \phi_0|t = 0)$, where $\theta(t)$ and $\phi(t)$ are the solution of the following classical equations of motion (with the initial conditions $\theta(t = 0) = \theta_0$, $\phi(t = 0) = \phi_0$):

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{1}{\sin\theta}\frac{\partial W^{(-1)}_{H_{\text{int}}}}{\partial\phi} = \sin\phi\sqrt{S\cos\theta + S + 1} \tag{10.133}$$

where $\tau = 2Sgt$. From the conservation of energy,

$$S\cos\phi\sin\theta\sqrt{S + 1 + S\cos\theta} = C \tag{10.134}$$

it follows that

$$\sin\phi = \sqrt{1 - \frac{C^2}{S^2\sin^2\theta(S\cos\theta + S + 1)}}$$

and we have the solution of the classical equations of motion in the form

$$\begin{aligned}\tau - \tau_0 &= \int_x^{x_2} \frac{\mathrm{d}x}{\sqrt{(1 - x^2)(x + 1 + 1/S) - C^2/S^3}}\\ &= \int_x^{x_2} \frac{\mathrm{d}x}{\sqrt{(x_2 - x)\,(x - x_{1+})\,(x - x_{1-})}}\end{aligned}$$

where $x = \cos\theta$, and the turning points $x_{1-} < x_{1+} < x_2 \leq 1$ are the roots of the cubic equation

$$\left(1 - x^2\right)(x + 1 + 1/S) - C^2/S^3 = 0 \tag{10.135}$$

so that

$$\frac{2}{\sqrt{x_2 - x_{1-}}} F\left(\arcsin\sqrt{\frac{x_2 - x}{x_2 - x_{1+}}}, \sqrt{\frac{x_2 - x_{1+}}{x_2 - x_{1-}}}\right) = \tau - \tau_0 \tag{10.136}$$

where $F(\psi, \kappa)$ is the elliptic integral of first kind. Inverting Equation 10.136 we may, in principle, determine the classical trajectories for $\theta(t)$ and $\phi(t)$.

The period of classical motion

$$2\int_{x_{1+}}^{x_2} \frac{\mathrm{d}x}{\sqrt{\left(1 - x^2\right)(x + 1 + 1/S) - C^2}}$$
$$= 2\int_{x_{1+}}^{x_2} \frac{\mathrm{d}x}{\sqrt{(x_2 - x)(x - x_{1+})(x - x_{1-})}} = T$$

also provides information about the spectral density of states in the large S limit:

$$\Delta E = \frac{2\pi}{T(E)} \tag{10.137}$$

Where, in the quantum domain, we may write: $\Delta E = E_{k+1} - E_k$, $T(E) = T((E_{k+1} + E_k)/2)$. Here, ΔE corresponds to a quantum transition frequency and determines the number of quantum states in the unit interval of energy.

For the roots of the cubic Equation 10.135, which determines the classical turning points, we can find the following approximation:

$$x_{1\pm} \approx -1 + \frac{1}{2S}\left(-1 \pm \sqrt{1 + 2C^2/S}\right), \quad x_2 \approx 1 - \frac{C^2}{4S^3} \tag{10.138}$$

We observe that when the initial energy is close to its minimum value, $C \approx -S^{3/2}\sqrt{32/27}$, the roots are well separated

$$x_{1-} < -1 < x_{1+} < x_2 < 1$$

and thus, the classical dynamic is quasi-regular, with the oscillation period

$$\frac{4}{\sqrt{x_2 - x_{1-}}} K\left(\sqrt{\frac{x_2 - x_{1+}}{x_2 - x_{1-}}}\right) = T \tag{10.139}$$

where $K(\kappa)$ is the complete elliptic integral of first kind. Nevertheless, in the region close to the separatrix, when the initial energy is close to zero, $C \sim 0$, the roots x_{1-} and x_{1+} become very close, $x_{1+} - x_{1-} \sim 1/S \ll 1$, which is reflected in

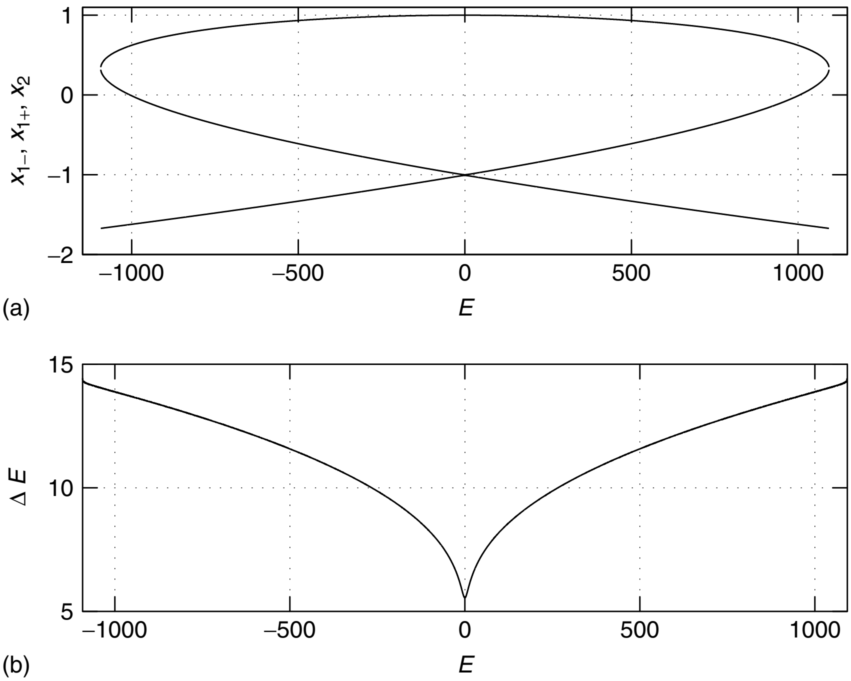

Figure 10.3 Quasi-classical dynamics of the Dicke model, S=100. (a) Classical turning points (roots of Equation 10.135) as functions of energy. (b) Oscillation frequency (Equation 10.137) as a function of energy.

the quasi-singular behavior of the transition frequencies at zero energy. We show the behavior of the classical turning points (exact roots of the cubic Equation 10.135 as functions of the initial energy C in Figure 10.3. The oscillation frequency (10.137) is also shown as a function of the energy. The singular behavior of the oscillation frequency at zero energy can be noted. These semiclassical results are in a good correspondence with the results obtained by the numerical diagonalization of the Dicke-model Hamiltonian.

Using the approximation (10.138) for the turning points, one can find the asymptotic for the oscillation period (Equation 10.139):

$$T \approx -\frac{1}{\sqrt{2}} \ln\left(\frac{1}{S^2} + \frac{2C^2}{S^3}\right)$$

Thus a logarithmic behavior arises, leading to a high density of energy spectrum at the region close to zero energy. The energy spectrum density becomes divergent in the classical limit.

10.5
Problems

10.1 Prove that

$$W_{S_j}(\theta,\phi)=\sqrt{S(S+1)}n_j \tag{10.140}$$

$$W_{\frac{1}{2}\left\{S_j,S_k\right\}_+}(\theta,\phi)=\frac{1}{2}\sqrt{(2S+3)(2S-1)S(S+1)}n_j n_k,\quad j\neq k \tag{10.141}$$

$$W_{S_j^2}(\theta,\phi)=\frac{1}{2}\sqrt{(2S+3)(2S-1)S(S+1)}\left(n_j^2-\frac{1}{3}\right)+\frac{S(S+1)}{3} \tag{10.142}$$

where n_j are the components of the unitary vector,

$$\mathbf{n}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$$

10.2 Prove that the Wigner function of an eigenstate, $|k,S\rangle$ of the operator S_z, has the form

$$W_k(\theta,\phi)=\frac{2\sqrt{\pi}}{\sqrt{2S+1}}\sum_{L=0}^{2S}\sqrt{\frac{2L+1}{2S+1}}Y^*_{L,0}(\theta,\phi)C^{Sk}_{Sk,\,L0}$$

10.3 Prove that the Stratonovich–Weyl symbol of a superposition of atomic coherent states (the Schrödinger cat state on the sphere)

$$|\psi\rangle=\frac{1}{\sqrt{2}}\left(|\theta=\frac{\pi}{2},\phi=0\rangle+|\theta=\frac{\pi}{2},\phi=\pi\rangle\right)$$

in the limit $S\gg 1$, has the form

$$W(\theta,\phi|t)=f(\theta,\phi|t)^{2S}-\mathrm{Re}\left(f(\theta,\phi+\pi/2|t)+i\cos\theta\right)^{2S}$$

where $f(\theta,\phi)=n_x$

10.4 Find the semiclassical evolution equation for the Hamiltonian

$$H=\chi\left(S_+^2+S_-^2\right)$$

and solve it.

10.5 Prove that the asymptotic form of the operators $\Lambda_{0,\pm}^{(0)}(\phi, \theta)$ in the semiclassical limit, $S \gg 1$, is given as follows:

$$\Lambda_0^{(0)}(\theta, \phi) \approx \frac{1}{2\varepsilon}\cos\theta - \frac{\varepsilon}{4}\left(\cos\theta + \sin\theta\partial_\theta + \cos\theta\mathcal{L}^2\right)$$

$$\Lambda_\pm^{(0)}(\theta, \phi) \approx \mathrm{e}^{\pm i\phi}\frac{\sin\theta}{4\varepsilon} \pm \frac{\varepsilon}{4}\left[2\cos\theta l_\pm - \mathrm{e}^{\pm i\phi}\sin\theta\left(2l_z \pm 1 \pm \mathcal{L}^2\right)\right]$$

10.6 Prove the following representations for the operator : $\hat{w}(\alpha, s)$

$$\hat{w}^{(s)}(\alpha) = \mathrm{e}^{-\frac{2}{1-s}|\alpha|^2}\,\mathrm{e}^{\frac{2}{1-s}\alpha a^\dagger}\,T(s)\,\mathrm{e}^{\frac{2}{1-s}\alpha^* a}$$

$$\hat{w}^{(s)}(\alpha) = \mathrm{e}^{\frac{2}{1+s}|\alpha|^2}\,\mathrm{e}^{-\frac{2}{1+s}\alpha^* a}\,T(s)\,\mathrm{e}^{-\frac{2}{1+s}\alpha a^\dagger}$$

$$\hat{w}^{(s)}(\alpha) = \mathrm{e}^{-\frac{2}{1-s}|\alpha|^2}\,T(s)\,\mathrm{e}^{\frac{-2}{1+s}\alpha a^\dagger}\,\mathrm{e}^{\frac{2}{1-s}\alpha^* a}$$

10.7 Prove that in the limit $N \gg 1$, the Q function of a number state $|N\rangle$ is approximated as

$$Q(\alpha) \approx \frac{1}{\sqrt{2\pi|\alpha|}}\exp\left[-\frac{(N - |\alpha|^2)^2}{2|\alpha|^2}\right]$$

10.8 Prove that the P function for the number state $|1\rangle$ is of the form

$$P(\alpha) = \delta^{(2)}(\alpha) + \partial_{\alpha^*}\partial_\alpha\delta^{(2)}(\alpha)$$

10.9 Find the Q function of the squeezed state

$$S(\eta)|\alpha\rangle, \quad \eta = r\,\mathrm{e}^{\mathrm{i}\theta}$$

and analyze it in the limit $|\alpha| \gg 1$.

11
Appendices

11.1
Lie Groups and Lie Algebras

11.1.1
Groups: Basic Concepts

In this appendix, we give an overview of some basic facts about group theory and the theory of Lie algebras [70, 122], which are used throughout the book (without attempting to approach these with mathematical rigor and rather appealing to the intuition of the reader). We begin with the definition of a group.

A set of elements $G = \{g_1, g_2, \ldots\}$ is called a *group* if it has the following properties:

1. For any of the elements g_1, g_2, $g_3 \in G$, a composition law $g_1 \cdot g_2 \in G$ is defined such that $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$.
2. There is an identity element e, such that $g \cdot e = e \cdot g = g$.
3. For every $g \in G$ there is an inverse element $g^{-1} \in G$, such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

In general, the elements of a group do not commute with each other: $g_1 \cdot g_2 \neq g_2 \cdot g_1$. If, however, $g_1 \cdot g_2 = g_2 \cdot g_1$ for every pair of elements $g_1, g_2 \in G$, the group is called *Abelian*.
The center of a group is the set of elements of the group that commute with every element of this group.
A subset H of the group G that by itself is a group is called a *subgroup* of G.
A subgroup $H \subset G$ is called *invariant* if for each $h \in H$ and any $g \in G$, the relation $ghg^{-1} \subset H$ is satisfied.
A group is called *simple* if it has no invariant subgroups.
A group is called *semisimple* if it has no invariant Abelian subgroups.
The right coset $C_{\mathrm{R}} = G/H$, where H is a subgroup of G is defined as a set of elements $C_{\mathrm{R}} = gH = \{gh : h \in H\}$ so that any element $g \in G$ can be represented as $g = c_k h_j$, where $h_j \in H$ and $c_k \in C_{\mathrm{R}}$. The element c_k, which is called the *right coset* representative, is not unique.

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Likewise, the *left coset spaces* $C_{\mathrm{L}} = H\backslash G$ is defined as $C_{\mathrm{L}} = Hg = \{hg : h \in H\}$, such that any element of the group $g \in G$ can be represented in the form $g = h_j \tilde{c}_k$, where $h_j \in H$ and $\tilde{c}_k \in C_{\mathrm{L}}$.

If the subgroup H is invariant, the right and left coset spaces coincide and have the structure of a group, called the *factor-group* G/H.

The set of all nonsingular linear transformations of an n-dimensional vector space forms a group denoted by $GL(n, R)$ or $GL(n, C)$, depending on whether entries can be complex or are restricted to be real. The notation $GL(n, C)$ stands for general linear group of transformations of the n-dimensional vector space over the complex field.

The group $GL(n,C)$ has several subgroups that are important in physics. The subgroups of $GL(n,C)$ for which the determinant of the corresponding linear transformations is equal to 1 are called *special*. They are denoted by the prefix S, so that $SL(n,C)$ is the group of special (i.e. $\det g = 1$) linear transformations in the n-dimensional space over the complex field.

Within the $GL(n,R)$ and $GL(n,C)$ groups, there are various subgroups characterized by the conservation of some (invariant) properties of the vectors on which they act. The following are among these:

1. Orthogonal groups $O(p, q)$, $p + q = n$: These are groups of transformation of a real n-dimensional vector space, which preserve the quadratic form $x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots x_{p+q}^2$.
2. Unitary groups $U(p, q)$ or $su(p, q)$: These are groups of transformations of a complex n-dimensional vector space, which preserve the quadratic form $|x_1|^2 + \cdots + |x_p|^2 - |x_{p+1}|^2 - ..|x_{p+q}|^2$.

If group elements are parametrized by a set of continuous parameters, the corresponding group is called *continuous*. $GL(n, R)$, $SL(n, R)$, $O(p, q)$, $U(p, q)$, and $su(p, q)$ are all examples of continuous groups.

The continuous groups that depend analytically on the parameters are called *Lie groups*. In such cases, an element of the group depending on n parameters is usually written as $g\,(\alpha_1, \alpha_2, \ldots, \alpha_n) \in G$.

The parameterization is often chosen so that, in the neighborhood of the identity, g is written as the exponential

$$g\,(\alpha_1, \alpha_2, \ldots, \alpha_n) = \exp\left(-\mathrm{i}\sum_{k=1}^{n} \alpha_k L_k\right) \tag{11.1}$$

where the operators L_k

$$L_k = \mathrm{i}\frac{\partial g\,(\alpha_1, \alpha_2, \ldots, \alpha_n)}{\partial \alpha_k}|_{\alpha=0}$$

are called (infinitesimal) *generators* of the group. The infinitesimal generators form the Lie algebra associated with the Lie group under discussion. Lie algebras are usually denoted by lower case letters, for instance, $gl(n, C)$ is the Lie algebra of the group $GL(n, C)$. Likewise, $su(p, q)$ is the Lie algebra of the group $su(p, q)$.

The maximum number of linearly independent generators of the group that commute with each other determine the *rank* of the group.

A Lie group is said to be *compact* if every element is parameterized by a finite number of bounded parameters. Otherwise, it is called a *noncompact* group.

Elements C_k commuting with each other and with every generator in the algebra are called *Casimir elements*. In a semisimple group of rank k there are k different Casimir elements.

11.1.2
Group Representations

The correspondence that assigns to each element $g \in G$ a linear operator that acts in a vector space V: $g \to T(g)$, such that

$$T(g_2)T(g_1) = T(g_2 g_1)$$

is called a *representation* of the group G over the space V (representation space). In this case, we have

1. $T(e) = I$
2. $T(g^{-1}) = T^{-1}(g)$
3. $\det T(g) \neq 0$

The Casimir element in the representation space V is put in correspondence with a Casimir operator $\hat{C}$.

The representation is called *unitary* if the operators $T(g)$ conserve the scalar product in the space V:

$$\big(T(g)x, T(g)y\big) = \big(x, y\big)$$

for any $x, y \in V$ and $g \in G$. In this case,

$$T^{-1}(g) = T(g^{-1})$$

is satisfied. For compact groups the property $T^{\dagger}(g) = T^{-1}(g) = T(g^{-1})$ is also fulfilled. Two representations $T_1(g)$ and $T_2(g)$ that act on the spaces V_1 and V_2 are called *equivalent* if there is an operator S, such that $ST_1(g) = T_2(g)S$, or $T_2(g) = ST_1(g)S^{-1}$.

The subspace W of the representation space V is called *invariant* if $T(g)W \subset W$ for any $g \in G$ and any vector in W.

A representation $T(g)$ is called *reducible* if the representation space V contains invariant subspaces. Otherwise, the representation is called *irreducible*.

The following irreducibility criterion (*Schur's Lemma*) is widely used: Let $T(g)$ be a unitary representation in the space V, then $T(g)$ is irreducible if any (bounded) linear operator, which commutes with all elements of the group, is proportional to the identity operator. Thus, the Casimir operators are proportional to the identity operator: $\hat{C}_k = \lambda_k I$ in an irreducible unitary representation. This property allows all the irreducible representations to be uniquely determined and classified by the set of eigenvalues of the Casimir operators λ_k to be classified.

All the irreducible unitary representations of a compact group are finite dimensional. All the irreducible unitary representations (except unidimensional representations) of a noncompact group are infinite dimensional.

11.1.3
Lie Algebras

The vector space $\mathcal{L}$ is called a *Lie algebra* if for any $L_k, L_j \in \mathcal{L}$ a bilinear operation, called the *Lie bracket*, is defined such that, $(L_k, L_j) \to [L_k, L_j] \in \mathcal{L}$ satisfies the following relations:

1. $[L_k, L_j] = -[L_j, L_k]$
2. $[L_k, aL_j + bL_m] = a[L_k, L_j] + b[L_k, L_m]$
3. $[L_k, [L_j, L_m]] + [L_m, [L_k, L_j] + [L_j, [L_m, L_k] = 0$ (Jacobi identity)

All these properties are automatically satisfied if the Lie bracket has the form of a commutator: $[L_k, L_j] = L_k L_j - L_j L_k$.

Since a Lie algebra $\mathcal{L}$ is a linear space, a basis $\{E_j\}$ can be chosen in this space, such that

$$[E_k, E_j] = \sum_m c_{kjm} E_m$$

where c_{kjm} are called *structure constants* and satisfy the conditions

- $c_{kjm} = -c_{jkm}$, (antisymmetry);
- $\sum_m c_{kjm} c_{mpq} + \sum_m c_{jpm} c_{mkq} + \sum_m c_{pkm} c_{mjq} = 0$, (a consequence of the Jacobi identity).

The structure constants completely determine the structure of the algebra. From the structure constants, the so-called metric tensor can be constructed

$$g_{jk} = \sum_{l,m} c_{jlm} c_{kml}$$

From the basis elements $\{E_j\}$, a set of monomials of the type

$$\{E_j,\ E_j E_k,\ E_j E_k E_l, \ldots\}$$

inequivalent under commutation, can be constructed. This set has the structure of a Lie algebra and is called the *universal enveloping algebra*.

A set of linearly independent generators of a Lie group span the corresponding Lie algebra. On the other hand, the elements $\exp\left(-\mathrm{i}\sum_k \alpha_k L_k\right)$, where L_k are generators of a Lie algebra, form the corresponding Lie group.

The Lie algebra is said to be commutative if $[L_k, L_j] = 0$, for all $L_k, L_j \in \mathcal{L}$.

A subspace $\mathcal{M} \subset \mathcal{L}$ is called a *subalgebra* of $\mathcal{L}$ if for any $L_k, L_j \in \mathcal{M}$, $[L_k, L_j] \in \mathcal{M}$ is satisfied.

The center of the algebra, denoted by Z, is formed by the elements of the algebra that commute with all the elements of the algebra.

The Casimir operators are the elements of the universal enveloping algebra although they are not elements of the algebra itself. A quadratic Casimir element is always given by

$$C_2 = g^{kj} L_k L_j$$

where g^{kj} are elements of the matrix that corresponds to the inverse metric tensor $g^{kj} = (g^{-1})_{kj}$.

The subalgebra $\mathcal{M} \subset \mathcal{L}$ is called an *ideal* of $\mathcal{L}$ if $[\mathcal{M}, \mathcal{L}] \subset \mathcal{M}$.

The Lie algebra $\mathcal{L}$ is called *simple* (or semisimple) if it does not contain an ideal (or commutative ideal). An algebra is semisimple if $\det g_{kj} \neq 0$ (*Cartan criterion*). The largest commutative subalgebra of a semisimple algebra is called the *Cartan subalgebra*; the dimension of the Cartan subalgebra is the rank of this algebra.

We define the sequence $\mathcal{L}_0 = \mathcal{L},\ \mathcal{L}_k = [\mathcal{L}_{k-1}, \mathcal{L}_{k-1}]$, so that each $\mathcal{L}_k$ is ideal in $\mathcal{L}$. The $\mathcal{L}$ algebra is said to be *soluble* if there is a number k_0 such that $\mathcal{L}_k = 0$, for all $k > k_0$.

We define another sequence $\mathcal{L}_0 = \mathcal{L}$, $\mathcal{L}_k = [\mathcal{L}_{k-1}, \mathcal{L}]$, so that each $\mathcal{L}_k$ is also ideal in $\mathcal{L}$. The $\mathcal{L}$ algebra is called a *nilpotent* if there is a number k_0 such that $\mathcal{L}_k = 0$, for all $k > k_0$. Thus, a nilpotent algebra is soluble.

The correspondence that assigns a linear operator that acts on a vector space V: $L_k \to T(L_k)$, to each element $L_k \in \mathcal{L}$ such that

1. $T(L_k) + T(L_j) = T(L_k + L_j)$
2. $T(aL_k) = aT(L_k)$
3. $T([L_k, L_j]) = [T(L_k), T(L_j)]$

is called a *representation of the* $\mathcal{L}$ algebra in the space V (representation space), and obviously the relation

$$[T(E_k), T(E_j)] = \sum_m c_{kjm} T(E_m)$$

is fulfilled.

The concepts of reducibility and irreducibility for Lie algebras are the same as the corresponding concepts for the groups. To find the representation of the algebra in a space V means to determine the matrix elements of the generators of the algebra in this space. In each invariant subspace with certain values of the Casimir operators, the basis elements can be chosen as eigenstates of the operators of the Cartan subalgebra.

For a Lie algebra, one can always construct a so-called adjoint representation, whose dimension is equal to the dimension of the algebra. In this representation, each generator (element of the algebra basis) is represented by a matrix $C^{(j)}$ whose elements are the structure constants:

$$E_j \to (C^{(j)})_{kl} = c_{jkl}$$

The matrixes $C^{(j)}$ satisfy all the properties of a representation, for example,

$$[C^{(j)}, C^{(k)}]_{lm} = \sum_n c_{jkn}(C^{(n)})_{lm}$$

11.1.4
Examples

Before proceeding to consider concrete algebras, let us overview a special case where two operators L and M form the so-called ladder algebra:

$$[L, M] = L \tag{11.2}$$

Let $|m\rangle$ be the eigenstate of M with the eigenvalue m

$$M|m\rangle = m|m\rangle$$

then, we have from Equation 11.2,

$$[L, M]|m\rangle = L|m\rangle = (mL - ML)\,|m\rangle$$

and the state $L|m\rangle$ is an eigenstate M with the eigenvalue $m-1$:

$$ML|m\rangle = (m-1)\;L|m\rangle$$

Thus, the operator L is a down-ladder operator, with respect to the eigenstates of the operator M. The consecutive application of operator L constructs an eigenstate ladder for the operator M with equidistant eigenvalues $L^n|m\rangle \sim |m-n\rangle$. If the operator M is Hermitian, the operator $L^\dagger$ will play the role of a raising operator: $[L^\dagger, M] = -L^\dagger,\ L^{\dagger n}|m\rangle \sim |m+n\rangle$. If the spectrum of the operator M is bounded from below (or from above), the application of the operator L ($L^\dagger$) to this state yield zero: $L|m_{\text{min}}\rangle = 0$ ($L^\dagger|m_{\text{max}}\rangle = 0$).

When $[L, M] = \mu L$, the following disentangling formula is useful in applications

$$e^{L+M} = \exp\left[L\frac{1-e^{-\mu}}{\mu}\right]\exp(M)$$

1. Heisenberg–Weyl Group $H(1)$ and Algebra $h(1)$

The group $H(1)$ is a noncompact Lie group. It is also non-semisimple, as is shown later. Group elements depend on three parameters $(s; x, y)$ and satisfy the composition rule:

$$g(s_1; x_1, y_1)g(s_2; x_2, y_2) = g(s_1 + s_2 + x_1y_2 - y_1x_2;\ x_1 + x_2,\ y_1 + y_2)$$

Nonunitary representations of this group include representations by triangular matrices in finite dimensions. The center of the group is formed by the elements $g(s; 0, 0)$. Besides the center, the group $H(1)$ has two Abelian subgroups formed by the elements $g(s; x, 0)$ and $g(s; 0, y)$. The group elements can be represented in the form (11.1) as follows:

$$g(s;\, x, y) = \exp[\mathrm{i}sI]\, D(\alpha)\,, \quad D(\alpha) = \exp\left[\alpha a^\dagger - \alpha^* a\right] \tag{11.3}$$

where $\alpha = (-x + \mathrm{i}y)/\sqrt{2}$. The generators $a^\dagger$, a and I satisfy the following commutation relations:

$$[a, a^\dagger] = I, \quad [a, I] = [a^\dagger, I] = 0 \tag{11.4}$$

and form the $h(1)$ algebra, which has two abelian subalgebras $\{a, I\}$ and $\{a^\dagger, I\}$, which are simultaneously (abelian) ideals. Thus, the algebra $h(1)$ is neither simple nor semisimple.

To construct a unitary representation of the $h(1)$ algebra, it is convenient to introduce a new Hermitian operator $\hat{n} = a^\dagger a$. The generators of the extended algebra, $a^\dagger a$, $a^\dagger$, a, and I satisfy the following commutation relations in addition to (11.4):

$$[\hat{n}, a^\dagger] = a^\dagger, \;\; [\hat{n}, a] = -a, \;\; [\hat{n}, I] = 0 \tag{11.5}$$

The eigenstates of $\hat{n}$ form an orthonormal basis in an infinite-dimensional Hilbert space $\mathcal{H}$

$$\hat{n}|n\rangle = \omega_n|n\rangle, \quad \langle n|k\rangle = \delta_{kn}$$

and satisfy the following resolution of the identity:

$$\sum_n |n\rangle\langle n| = I$$

Besides, the operator $\hat{n}$ is positively defined, so its average over any state in $\mathcal{H}$ is nonnegative:

$$\langle\psi|\hat{n}|\psi\rangle = \sum_n \langle\psi|a^\dagger|n\rangle\langle n|a|\psi\rangle = \sum_n |\langle n|a|\psi\rangle|^2 \geq 0, \;\; |\psi\rangle \in \mathcal{H}$$

and is equal to zero on the state $|0\rangle$, which satisfies the condition

$$a|0\rangle = 0$$

In a view of Equation 11.5, the operators $(\hat{n}, a)$ and $(\hat{n}, a^\dagger)$ form two ladder subalgebras: raising and lowering. The state $|0\rangle = |n{=}0\rangle$ is the lowest state ("vacuum state") in the space of the eigenstates of the operator n. From this state, the whole basis can be constructed by applying the raising operator $a^\dagger$. The spectrum of operators $\hat{n}$ is the entire set of nonnegative values, $n = 0, 1, 2, \ldots$. The application of the operators $a^\dagger$ and a (commonly called *creation* and *annihilation operators*) to the basis elements can be found with the standard techniques. It is clear that

$$a|n\rangle = \lambda_n|n-1\rangle, \quad a^\dagger|n\rangle = \widetilde{\lambda}_n|n+1\rangle$$

thus,

$$\lambda_n = \langle n-1|a|n\rangle, \quad \widetilde{\lambda}_n = \langle n+1|a^\dagger|n\rangle = \langle n|a|n+1\rangle^* = \lambda^*_{n+1}$$

and, $\lambda_0 = 0$. Since

$$\langle n|\hat{n}|n\rangle = n = \langle n|a^\dagger|n-1\rangle\langle n-1|a|n\rangle = |\lambda_n|^2$$

we have $\lambda_n = \sqrt{n}$, which determines the matrix elements of the operator in the representation space. Finally, the basis in the representation space is constructed as

$$|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle$$

The following ordering formulae hold:

$$\begin{aligned}
\left(a + a^{\dagger}\right)^{n} &= \sum_{k=0}^{[n/2]} \sum_{l=0}^{n-2k} \left(-\frac{1}{2}\right)^{k} \frac{n!}{k!l!\left(n-2k-l\right)!} a^{l} a^{\dagger n-2k-l} \\
&= \sum_{k=0}^{[n/2]} \sum_{l=0}^{n-2k} \frac{1}{2^{k}} \frac{n!}{k!l!\left(n-2k-l\right)!} a^{\dagger l} a^{n-2k-l} \\
a^{m} a^{\dagger n} &= \sum_{k=0}^{\min(m,n)} \frac{n!m!}{k!\left(n-k\right)!\left(m-k\right)!} a^{\dagger n-k} a^{m-k} \\
a^{\dagger m} a^{n} &= \sum_{k=0}^{\min(m,n)} \frac{(-1)^{k}\, n!m!}{k!\left(n-k\right)!\left(m-k\right)!} a^{n-k} a^{\dagger m-k} \\
\exp\left(\mu a^{\dagger} a\right) &= \sum_{k=0}^{\infty} \frac{(\mathrm{e}^{\mu} - 1)^{k}}{k!} a^{\dagger k} a^{k}
\end{aligned}$$

2. The *su*(2) Algebra and the *su*(2) Group

Let us consider the group of 2 × 2 unitary matrices with unit determinant [123]:

$$g = \begin{bmatrix} a & b \\ -b^{*} & a^{*} \end{bmatrix}, \quad |a|^{2} + |b|^{2} = 1 \tag{11.6}$$

where $a = a_1 + \mathrm{i}a_2$ $b = b_1 + \mathrm{i}b_2$ (a_1, a_2, b_1, b_2 are real numbers and a^{*} stands for complex conjugation). The group manifold is a three-dimensional sphere, and thus the group is compact. The elements of the center are formed by the matrixes I and $-I$. The generators of the group can be chosen as Hermitian operators $S_x, S_{y,} S_z$, where $S = \boldsymbol{\sigma}/2$ and $\sigma_j,\ j = x, y, z$ are the Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

These matrices satisfy the following properties:

1. Commutation relations:

$$[\sigma_k, \sigma_j] = 2\mathrm{i}\varepsilon_{kjl}\sigma_l$$

2. Pauli matrix algebra:

$$\sigma_{x,y,z}^{2} = I, \quad \sigma_k\sigma_j = \delta_{ik} + \mathrm{i}\varepsilon_{kjl}\sigma_l$$

where ε_{kjl} is the Levi–Civita antisymmetrical tensor.

3. Any function of the matrixes σ_l is reduced to a linear polynomial:

$$f(\boldsymbol{\sigma}) = a + (\mathbf{b}\boldsymbol{\sigma}), \quad a = \frac{1}{2}\mathrm{Tr}\left(f\right), \quad b = \frac{1}{2}\mathrm{Tr}\left(\boldsymbol{\sigma} f\right)$$

   where $\boldsymbol{\sigma}$ is a vector constituted by the matrices σ_x, σ_y, σ_z.

4. Decomposition of the exponential: This is a special case of the above property given by

$$\exp\left[\mathrm{i}\phi\left(\boldsymbol{\sigma}\mathbf{n}\right)\right] = \cos\phi + \mathrm{i}\left(\boldsymbol{\sigma}\mathbf{n}\right)\sin\phi$$

From the operators S_x, S_y, S_z, the non-Hermitian raising and lowering operators $S_\pm$ can be constructed

$$S_\pm = S_x \pm \mathrm{i}S_y$$

which satisfy the following commutation relations:

$$[S_z, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = 2S_z$$

The operators S_x, S_y, S_z form the basis of the *su*(2) algebra. This algebra is simple (and thus is semisimple) and the metric tensor is proportional to the Kroneker symbol: $g_{kj} = 2\delta_{kj}$. The Cartan subalgebra contains only one element, which is traditionally chosen to be S_z; thus, the *su*(2) algebra is of rank 1. The Casimir operator is given by

$$C_2 = S_x^2 + S_y^2 + S_z^2 = S_z^2 + \frac{1}{2}(S_+S_- + S_-S_+)$$

Any unitary irreducible representation of the *su*(2) algebra is characterized by a non-negative integer or a half-integer number $S = 0,\ 1/2,\ 1,\ 3/2, \ldots$; the representation dimension is $N = 2S + 1$ and the value of the Casimir operator is $(1 + N/2)\,N/2$. Within each irreducible representation subspace, the basis elements can be chosen as eigenstates of the Cartan element S_z. Since the pairs (S_+, S_z) and (S_-, S_z) form raising and lowering ladder subalgebras, the operator S_z has an equidistant spectrum

$$S_z|k, N\rangle = \left(k - \frac{N}{2}\right)|k, N\rangle, \quad \langle m, N|k, N\rangle = \delta_{mk}, \;\; k = 0, 1, \ldots, N \qquad (11.7)$$

and the following resolution of the identity takes place:

$$\sum_{k=0}^{N} |k, N\rangle\langle k, N| = I$$

The representation of the *su*(2) algebra in each subspace is constructed in the same way as was done for the oscillatory algebra and the matrix elements of the operators $S_\pm$ are

$$S_+|k, N\rangle = \sqrt{(k+1)(N-k)}|k+1, N\rangle$$

$$S_-|k, N\rangle = \sqrt{k(N-k+1)}|k, N\rangle$$

so that

$$|k, N\rangle = \sqrt{\frac{k!(N-k)!}{N!}}\, S_+^k |0, N\rangle, \quad S_-|0, N\rangle = 0$$

In terms of the operators S_x, S_y, and S_z, an arbitrary element of the *su*(2) group can be parameterized in the following ways:

1. Euler parameterization:

$$g = \mathrm{e}^{-\mathrm{i}\varphi S_z}\mathrm{e}^{-\mathrm{i}\theta S_y}\mathrm{e}^{-\mathrm{i}\psi S_z} \tag{11.8}$$

   where the Euler angles take values in the intervals

$$0 \le \theta < \pi, \quad -2\pi \le \psi < 2\pi, \quad 0 \le \varphi < 2\pi$$

2. Polar parameterization:

$$g = \mathrm{e}^{-\mathrm{i}\omega(S\cdot\mathbf{n})} \tag{11.9}$$

   where $S = \left(S_x, S_y, S_z\right)$ is the operator vector and $n = (\sin\Theta\cos\Phi,\ \sin\Theta\sin\Phi,\ \cos\Theta)$ is the unit vector that indicates the direction of the rotation axis with the angle ω.

3. Gauss (normal) parameterization:

$$g = \mathrm{e}^{aS_+}\mathrm{e}^{bS_z}\mathrm{e}^{cS_-} \tag{11.10}$$

4. Cayley–Klein parameterization: The parameters here are the matrix elements a and b of the two-dimensional representation (11.6); they are related to the Euler angles as

$$g = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}(\varphi+\psi)/2}\cos(\theta/2) & -\mathrm{e}^{-\mathrm{i}(\varphi-\psi)/2}\sin(\theta/2) \\ \mathrm{e}^{\mathrm{i}(\varphi-\psi)/2}\sin(\theta/2) & \mathrm{e}^{\mathrm{i}(\varphi+\psi)/2}\cos(\theta/2) \end{bmatrix}$$

The *su*(2) algebra can be realized in terms of boson annihilation and creation operators a and $a^\dagger$, $[a, a^\dagger] = 1$ (*the Holstein–Primakoff realization*) in the following way:

$$S_z = a^\dagger a - \frac{N}{2}, \quad S_+ = a^\dagger\sqrt{N - a^\dagger a}, \quad S_- = \sqrt{N - a^\dagger a}\,a$$

The representation space, in this case, is the space of the eigenstates of the operator $n = a^\dagger a$. However, the square roots that appear in the operators $S_\pm$ cut out of this infinite-dimensional space an $(N+1)$-dimensional subspace with the basis, $|0\rangle, |1\rangle, \ldots, |N\rangle$. This subspace is invariant in the sense that the action of the $su(2)$ representation generators $S_j, j = x, y, z$ on any element of this subspace does not take it outside this subspace.

Another realization frequently used in applications is *the Schwinger realization*. In this case, two types of operators $a^\dagger, a$ and $b^\dagger, b$ are introduced such that

$$[a, a^\dagger] = 1,\ [b, b^\dagger] = 1,\ \ [a, b] = [a^\dagger, b] = [a, b^\dagger] = [a^\dagger, b^\dagger] = 0$$

In terms of these operators, the generators of the $su(2)$ algebra representation can be written as

$$S_z = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right), \quad S_+ = a^\dagger b, \quad S_- = b^\dagger a$$

In this realization, besides the Casimir operator, there is another operator $\hat{N}$ that commutes with all the operators of the algebra:

$$\hat{N} = a^\dagger a + b^\dagger b$$

The representation space is the direct product of the spaces of two oscillatory algebras and the basis in this space is constituted by the eigenstates $|n_a, n_b\rangle$ of the operators $\hat{n}_a = a^\dagger a,\ \hat{n}_b = b^\dagger b$, and thus, of the operators $\hat{N}$ and S_z. The construction of the space begins from the lowest state $|0_a, 0_b\rangle$

$$a|0_a, 0_b\rangle = b|0_a, 0_b\rangle = 0$$

by applying the operators $a^\dagger$ and $b^\dagger$:

$$|n_a, n_b\rangle = \frac{a^{\dagger n_a} b^{\dagger n_b}}{\sqrt{n_a! n_b!}} |0_a, 0_b\rangle$$

It can easily be seen that

$$S_z|n_a, n_b\rangle = \frac{1}{2}(n_a - n_b)\,|n_a, n_b\rangle, \quad \hat{N}|n_a, n_b\rangle = (n_a + n_b)\,|n_a, n_b\rangle$$

The standard basis $|k, N\rangle$ (11.7) is constructed as

$$|k, N\rangle = \frac{(a^\dagger)^k\,(b^\dagger)^{N-k}}{\sqrt{k!\,(N-k)!}}|0, 0\rangle$$

The subspace with a fixed $N = n_a + n_b$ (eigenvalue of the operator $\hat{N}$) is invariant under the action of the generators of the $su(2)$ algebra. Obviously, the value of the Casimir operator in each invariant subspace is $(1 + N/2)\,N/2$.

3. *su*(*n*) Group and Algebra

The $U(n)$ group is a group of unitary matrixes in n dimensions. An element of the $U(n)$ group depends on n^2 real parameters and can always be represented in the form

$$U = \exp[\mathrm{i}H]$$

where H is a Hermitian matrix and Tr, $H = a$ is a real number. It follows from this that

$$|\det U| = 1$$

The $su(n) \subset U(n)$ group is a group of unitary $n \times n$ matrixes with the determinant equal to one. It is a compact group of rank $n - 1$. Any element $\widetilde{U}$ of the group $su(N)$ can be represented in the form

$$\widetilde{U} = \exp\left[\mathrm{i}\widetilde{H}\right], \ \mathrm{Tr}\,\widetilde{H} = 0$$

and depends on $n^2 - 1$ parameters. The generators of the $su(n)$ algebra are traceless, which guarantees that $\det\widetilde{U} = 1$, $\widetilde{U} \in su(n)$, since

$$\det\widetilde{U} = \exp\left[\mathrm{i}\mathrm{Tr}\,\widetilde{H}\right]$$

Obviously, any element of the group $U(n)$ can be represented in the following way:

$$U = \exp[\mathrm{i}H] = \exp\left[\mathrm{i}\left(\widetilde{H} + \frac{a}{n}I\right)\right] = \exp\left[\mathrm{i}\widetilde{H}\right]\exp\left[\mathrm{i}\frac{a}{n}I\right] \tag{11.11}$$

where I is the identity matrix in n dimensions and a is a real number, so that

$$\mathrm{Tr}, H = \mathrm{Tr}, \left(\widetilde{H} + \frac{a}{n}I\right) = \frac{a}{n}\,\mathrm{Tr}, I = a$$

Note that $\exp[\mathrm{i}\,I\,a/n]$ is an element of the group $U(1)$. Thus, from Eq. (11.11), we can immediately reach the conclusion that any element of the group $U(n)$ can be represented as a product of an element of the group $su(n)$ with an element of the group $U(1)$.

A basis of the $su(n)$ algebra is constituted by $n^2 - 1$ generators. It is convenient to have in mind the realizations of these generators by the $(n \times n)$-dimensional hermitian matrices (the n-dimensional representation of the $su(n)$). There are $n - 1$ generators that commute with each other forming the maximal commutative subalgebra of the $su(n)$ (Cartan subalgebra). The basis exists where these operators are simultaneously diagonal. Recall that Tr, $H_j = 0$. Let us denote by e_{ij}, the matrix that has the only unit matrix elements at the intersection of the ith row and the jth column, and all the other matrix elements are zeros. Then the n-dimensional realization of the Cartan subalgebra can be chosen as

$$H_j = e_{n-j,n-j} - e_{n-j+1,n-j+1}, \ \ j = 1, 2, \ldots, n-1$$

Now we may introduce $n - 1$ raising and lowering operators (“simple root vectors”) E_{+j}, E_{-j}, that correspond to the n-dimensional realization $E_{+j} = e_{n-j,n-j+1}$, $E_{-j} = e_{n-j+1,n-j}$. They have the following commutation relations with the operators from the Cartan subalgebra:

$$[H_j, H_k] = 0$$

$$\left[H_j, E_{\pm k}\right] = \pm\kappa_{j,k}E_{\pm k}$$

$$\left[E_{+j}, E_{-k}\right] = \delta_{jk}H_j$$

Here $\kappa_{j,k}$ is the so-called Cartan matrix, which for $su(n)$, has the form

$$\kappa = \begin{vmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{vmatrix}$$

The rest of the $(n-1) * (n-2)/2$ raising and lowering operators that correspond in the n-dimensional realization to the matrices $e_{j,k}$, $|k-j| > 1$ can now be obtained by repeating commutation of simple root vectors, $[E_{+m_1}, E_{+m_2}]$, $[E_{+m_1}, [E_{+m_2}, E_{+m_3}]], \dots$. This construction is called the Cartan–Weyl basis. The complete set of commutators for the su(3) algebra in the Cartan–Weyl basis can be found in Section 6.8 (Equations 6.51–6.53).

Representations of the $su(n)$ algebra that are important in applications can be obtained using the Schwinger method. Let us introduce n^2 operators $X_{\pm}^{jk}$ in the same way as we did for the $su(2)$ algebra:

$$X^{jk} = b_j b_k^{\dagger}, \quad k,j = 1, 2, \dots, n$$

where the operators $b_k^{\dagger}$ and b_j satisfy the commutation rules of the algebra $h(1)$:

$$[b_j, b_k^{\dagger}] = \delta_{jk}, \quad [b_k, b_j] = [b_k^{\dagger}, b_j^{\dagger}] = 0$$

and act on the space spanned by the vectors $|\{n_j\}_{j=1}^{n}\rangle$, eigenvectors of the operators $n_j = b_j^{\dagger} b_j$. The operators $X_{\pm}^{jk}$ constructed in this way satisfy the following commutation relations:

$$[X^{jk}, X^{nm}] = \delta_{nk} X^{jm} - \delta_{jm} X^{nk}$$

The diagonal generators X^{nn} form an Abelian subalgebra. Now, from the operators X^{nm} we can construct new operators S^{nm} with a zero trace by subtracting the operator $\sum_j X^{jj}$ (which is a constant in each invariant subspace $[\sum_j X^{jj}, X^{nk}] = 0$) from the diagonal operators

$$S_{+}^{jk} = X^{jk}, \quad S_{-}^{jk} = (S_{+}^{jk})^{\dagger}, \qquad k,j = 1, 2, \dots, n, \quad k > j$$

$$S^{nn} = X^{nn} - \frac{1}{n} \sum_j X^{jj}, \quad \sum_j S^{jj} = 0$$

and that satisfy the same commutation relations as the operators X^{jk}. The number of independent operators is $n^2 - 1$. Another way of constructing diagonal operators with a zero trace would be

$$S^{nn} = X^{nn} - X^{n-1\,n-1}$$

and we come back to the Cartan subalgebra.

The representation space $|n_1, n_2, \dots, n_n\rangle$ is constructed (as in the case of the $su(2)$ algebra) starting from the state $|0_1, 0_2, \dots, 0_n\rangle$ by applying powers of the operators

$b_j^\dagger$. The matrix elements of the generators of the $su(n)$ algebra in this basis are obvious, for example,

$$\begin{aligned}
&S_+^{jk}|n_1,\ldots,n_j,\ldots,n_k,\ldots,n_n\rangle \\
&\quad= \sqrt{n_j(n_k+1)}|n_1,\ldots,n_j-1,\ldots,n_k+1,\ldots,n_n\rangle \\
&S_-^{jk}|n_1,\ldots,n_j,\ldots,n_k,\ldots,n_n\rangle \\
&\quad= \sqrt{n_k(n_j+1)}|n_1,\ldots,n_j+1,\ldots,n_k-1,\ldots,n_n\rangle
\end{aligned}$$

4. The *su*(1,1) Group and Its Algebra

The group $su(1,1)$ is defined as the set of 2×2 matrices that satisfy the relation

$$gSg^\dagger = S$$

where g is the element of the group and the matrix S is defined as

$$S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The group element is represented by

$$g = \begin{bmatrix} u & v \\ v^* & u^* \end{bmatrix}, \quad |u|^2 - |v|^2 = 1$$

where $u = u_1 + iu_2,\ v = v_1 + iv_2$ (u_1, u_2, v_1, v_2 are real numbers). In terms of the Euler parameterization, an element of the $su(1,1)$ group has the form

$$g\,(\theta, \varphi, \psi) = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}(\varphi+\psi)/2}\cosh(\tau/2) & \mathrm{e}^{\mathrm{i}(\psi-\varphi)/2}\sinh(\tau/2) \\ \mathrm{e}^{-\mathrm{i}(\psi-\varphi)/2}\sinh(\tau/2) & \mathrm{e}^{\mathrm{i}(\varphi+\psi)/2}\cosh(\tau/2) \end{bmatrix}$$

where the Euler angles change in the intervals

$$0 \le \tau < \infty, \quad -2\pi \le \psi < 2\pi, \quad 0 \le \varphi < 2\pi$$

Considering the elements of the $su(1,1)$ group as transformations that act on the two-dimensional complex vector space $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ it can easily be seen that the group preserves the quadratic form $|z_1|^2 - |z_2|^2$:

$$g\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_1' \\ z_2' \end{bmatrix}, \quad |z_1|^2 - |z_2|^2 = |z_1'|^2 - |z_2'|^2$$

The group manifold is a hyperboloid, and thus it is noncompact. The general theory of noncompact groups establishes that the unitary representations are all infinite dimensional, with the exception of the trivial one-dimensional identity representation. The generators of the $su(1,1)$ group are operators $K_i,\ i = 0, 1, 2,$ that satisfy the commutation relations

$$[K_1, K_2] = -\mathrm{i}K_0, \quad [K_2, K_0] = \mathrm{i}K_1, \quad [K_0, K_1] = \mathrm{i}K_2$$

and form the $su(1,1)$ algebra. Introducing the raising and lowering operators $K_\pm$ in the standard form

$$K_\pm = K_1 \pm \mathrm{i}K_2, \quad K_+ = (K_-)^\dagger$$

we obtain

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0 \tag{11.12}$$

It can be easily seen that the second relation differs in sign from the corresponding commutation relation for $su(2)$. The Casimir element has the form

$$C_2 = K_0^2 - K_1^2 - K_2^2 = K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+), \quad [C_2, K_i] = 0$$

In the fundamental (nonunitary) representation the generators of the algebra have the form

$$K_+ = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad K_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad K_0 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The $su(1,1)$ algebra has different series of unitary representations, but we consider only the discrete series. A basis for an irreducible representation is given by the set $\{|\mu, k\rangle\}$, where k is the representation index. The action of the generators on basis vectors is

$$\begin{aligned} K_0|\mu, k\rangle &= \mu|\mu, k\rangle \\ K_-|\mu, k\rangle &= \nu_\mu|\mu - 1, k\rangle \\ K_+|\mu, k\rangle &= \widetilde{\nu}_\mu|\mu + 1, k\rangle, \quad \widetilde{\nu}_\mu = \nu^*_{\mu+1} \end{aligned}$$

The value of the Casimir operator in the invariant space $|\mu, k\rangle$ is equal to

$$C_2 = \mu^2 - |\nu_\mu|^2 - \mu = \mu^2 - |\widetilde{\nu}_\mu|^2 + \mu \tag{11.13}$$

Determining $\mu_{\min}$ from the condition $\nu_{\mu_{\min}} = 0$, we have

$$C_2 = \mu_{\min}^2 - \mu_{\min}$$

or, defining $k = \mu_{\min}$

$$C_2 = k(k-1)$$

so that $\mu = k + m,\ m = 0, 1, 2, \ldots.$ and

$$\begin{aligned} \nu_\mu &= \sqrt{m(m + 2k - 1)} \\ \widetilde{\nu}_\mu &= \sqrt{(m+1)(m+2k)} \end{aligned}$$

By construction, the representation k has a lowest state, with $\mu = k$, that satisfies $K_-|k, k\rangle = 0$; any other vector in the basis is constructed by applying the raising operator K_+:

$$|\mu = k + m, k\rangle \sim K_+^m|\mu, k\rangle$$

Alternatively, an irreducible representation with a highest vector, satisfying the condition $K_+|-k, k\rangle = 0$, can also be constructed:

$$|\mu = -k - m, k\rangle \sim K_-^m|-k, k\rangle$$

The value of the Casimir operator in this representation is $C_2 = k(k-1)$.

11.2
Coherent States

In this appendix, we briefly overview the concept of coherent states and we consider examples of coherent states for the groups $H(1)$ and $su(2)$. We begin with the standard definition of the coherent states for any Lie group [14, 126].

Let $T(g)$ be an irreducible unitary representation of the Lie group G, which acts on the Hilbert space H of the states of a quantum system:

$$|\psi\rangle \in \mathrm{H}, \quad T(g)|\psi\rangle = |\psi_g\rangle$$

The subgroup $H_0 \subset G$ is called a *stationary subgroup* of the state $|\psi_0\rangle \in \mathrm{H}$ if for any $h \in H_0$

$$T(h)|\psi_0\rangle = \mathrm{e}^{\mathrm{i}f(h)}|\psi_0\rangle$$

It is obvious that if H_0 is the stationary subgroup of the state $|\psi_0\rangle$, this state is an eigenstate for all the generators of the group H_0. The Lie algebra corresponding to the subgroup H_0 is called the *stationary* subalgebra for the state $|\psi_0\rangle$. Since the states of a quantum system are defined up to the multiplication by a global phase, any element of the orbit of the state $|\psi_0\rangle$, $|\psi_g\rangle = T(g)|\psi_0\rangle$, can be put in correspondence with an element of the coset space G/H_0.

At this stage, we need to fix, once and for all, a vector $|\psi_0\rangle$. This fixed vector is called the *fiducial*. The set of vectors $\{|\xi_g\rangle\}$ obtained by group transformation of the fiducial vector are a set of coherent states for the group with respect to the chosen fiducial vector.

Obviously, because any element of the subgroup H_0 returns $|\psi_0\rangle$ up to a phase, distinct coherent states are obtained when different elements in the coset G/H_0 act on $|\psi_0\rangle$:

$$|\xi_g\rangle = T(\widetilde{g})|\psi_0\rangle$$

where $\widetilde{g} \in G/H_0$.

The following properties are satisfied:

1. When applying the operator of a group representation to the state $|\psi_0\rangle$ a coherent state is generated (up to a phase):

$$T(g)|\psi_0\rangle = e^{if(g)}|\xi\rangle, \quad g \in G \tag{11.14}$$

2. The operators of the representation transform a coherent state into another:

$$T(g_1)|\xi\rangle = e^{i\beta(g,g_1)}|\xi_1\rangle \tag{11.15}$$

3. The set of coherent states form a basis (overcomplete) in the state space of the system H:

$$\int_{G/H} d\mu_g |\xi\rangle\langle\xi| = I \tag{11.16}$$

where μ_g is the invariant measure in the coset space G/H.

In principle, coherent states can be generated from any state $|\psi_0\rangle \in$ H. However, it has been shown that the states $|\psi_0\rangle$ that have the largest stationary subgroups generate coherent states closest to the classical states in the sense that they minimize the uncertainty relations and (in the case of the compact algebras) also the fluctuations of the quadratic Casimir operator.

An important class of coherent state systems corresponds to the coset spaces G/H that are homogeneous Kahlerian manifolds. Then, a natural symplectic structure can be introduced on the manifold G/H, so that it can be considered as a phase space of a classical dynamical system.

11.2.1
Examples

1. Heisenberg–Weyl Group $H(1)$

The representation space of the group (space of the states of the harmonic oscillator) is formed by the vectors

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$$

where $|0\rangle$ is the vacuum state, which is annihilated by the operator a: $a|0\rangle = 0$. First, we note that the state that has the largest stationary subgroup is the vacuum state, since the corresponding stationary subalgebra contains two elements (I, a). In other words, the operator that corresponds to the element of the group $(s; 0, 0)$ leads to the multiplication of the state $|0\rangle$ by a phase

$$T(s; 0, 0)|0\rangle = e^{is}|0\rangle$$

and the operator that corresponds to the element $(0, x, 0)$ leaves the state $|0\rangle$ invariant. A general element of the group has the standard form

$$T(g) = D(\alpha)\, e^{is}, \; D(\alpha) = \exp\left[\alpha a^\dagger - \alpha^* a\right]$$

According to the definition, the coherent states are generated by applying the displacement operator $D(\alpha)$ to the vacuum state $|0\rangle$ and different coherent states correspond to the different values of α:

$$D(\alpha)|0\rangle = |\alpha\rangle$$

Each coherent state $|\alpha\rangle$ is labeled by an element of the coset space $H(1)/H$, which is a complex plane $(\mathrm{Re}\,\alpha, \mathrm{Im}\,\alpha)$ and coincides with the phase space for the corresponding classical system. Introducing the operators

$$\hat{q} = \frac{a + a^\dagger}{\sqrt{2}}, \ \hat{p} = \frac{a - a^\dagger}{\mathrm{i}\sqrt{2}}$$

we obtain that their averages over the coherent states

$$\langle\alpha|\hat{q}|\alpha\rangle = \frac{\alpha + \alpha^*}{\sqrt{2}} = q, \ \langle\alpha|\hat{p}|\alpha\rangle = \frac{\alpha - \alpha^*}{\mathrm{i}\sqrt{2}} = p$$

with fluctuations given by

$$\sigma_q = \langle\alpha|\hat{q}^2|\alpha\rangle - \langle\alpha|\hat{q}|\alpha\rangle^2 = \frac{1}{2}$$

$$\sigma_p = \langle\alpha|p^2|\alpha\rangle - \langle\alpha|p|\alpha\rangle^2 = \frac{1}{2}$$

It is worth noting that the relation $\sigma_q = \sigma_p$, is invariant under rotation in the $(\mathrm{Re}\,\alpha,\ \mathrm{Im}\,\alpha)$ plane.

In terms of real-valued variables (q, p) the symplectic structure on the plane $\mathcal{R}^2$ is given as follows:

$$\{f, g\} = \partial_p f\, \partial_q g - \partial_q f\, \partial_p g$$

and the canonical coordinates (q, p), which satisfy $\{p, q\} = 1$, define the invariant 2-form $\omega = \mathrm{d}p \wedge \mathrm{d}q$. The symmetry group $H(1)$ acts on $\mathcal{R}^2$ as $(s; x, y)(q, p) = (q + y, p - x)$.

In the plane $(\mathrm{Re}\,\alpha,\ \mathrm{Im}\,\alpha)$, a coherent state can be represented as a circle with radius 1/2 centered at the point (x_0, p_0). The invariant measure in this space is

$$\mathrm{d}\mu = \frac{1}{\pi}\mathrm{d}^2\alpha = \frac{1}{\pi}\mathrm{d}\,(\mathrm{Re}\,\alpha)\ \mathrm{d}\,(\mathrm{Im}\,\alpha)$$

so that the resolution identity (11.16), reads

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha|\alpha\rangle\langle\alpha| = I$$

The coherent states $|\alpha\rangle$ form a nonorthogonal (overcomplete) basis

$$|\langle\alpha|\beta\rangle|^2 = \exp\left[|\alpha - \beta|^2\right]$$

in the representation space of the group.

Obviously, the operators $D(\alpha)$ do not form a group:

$$D(\alpha)D(\beta) = \exp\left[\mathrm{i}\,\mathrm{Im}\left(\alpha\beta^*\right)\right] D(\alpha+\beta)$$

They transform a coherent state into another $D(\alpha)|\beta\rangle = \exp\left[\mathrm{i}\,\mathrm{Im}\left(\alpha\beta^*\right)\right]|\alpha+\beta\rangle$. It is known that the coherent states $|\alpha\rangle$ are maximally classical, in the sense that they minimize the Heisenberg–Robertson uncertainty relation.

2. *su*(2) Group

Let us consider the unitary, irreducible representation of dimension $N+1$; the value of the Casimir operator is $(1+N/2)\,N/2$. The basis set, formed by the eigenstates of S_z operator, $|k,N\rangle$, $k=0,1,\ldots,N$ is constructed from the state $|0,N\rangle$ $(S_-|0,N\rangle = 0)$ by applying the raising operator S_+:

$$|k,N\rangle = \sqrt{\frac{(N-k)!}{k!N!}}\,S_+^k|0,N\rangle$$

The state $|0,N\rangle$ has a maximum stationary subalgebra, formed by the operators S_z and S_-. The elements $\exp(\mathrm{i}\psi S_z)$ form a subgroup $U(1)$ of the group $su(2)$ and are reduced to a phase when applied to the state $|0,N\rangle$:

$$\exp\left(\mathrm{i}\psi S_z\right)|0,N\rangle = \exp\left(\frac{-\mathrm{i}\psi N}{2}\right)|0,N\rangle$$

Note that any element of the $su(2)$ group can be represented in the form $g=\tilde{g}H$, where $H = U(1)$ is a subgroup of $su(2)$ formed by the diagonal matrices of the type

$$\begin{bmatrix} \exp\left(\dfrac{-\mathrm{i}\psi}{2}\right) & 0 \\ 0 & \exp\left(\dfrac{\mathrm{i}\psi}{2}\right) \end{bmatrix}$$

and $\tilde{g}$ are elements of the coset $su(2)/U(1)$ having the form

$$\begin{bmatrix} a & -b \\ b^* & a \end{bmatrix},\quad a^2+|b|^2 = 1,\ a>0$$

where

$$a = \cos\ (\theta/2),\ b = \mathrm{e}^{-\varphi/2}\sin\ (\theta/2)$$

From this, one can show that the quotient space $su(2)/U(1)$ is isomorphic to the sphere S^2. It is convenient to identify points of a unit sphere by vectors $\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. In fact, the sphere S^2 is the homogeneous space of the $su(2)$ group: the action of $su(2)$ elements maps points on the sphere to other points on the sphere, $g\cdot\mathbf{n} = \mathbf{n}' \in S^2$. In particular, $g(\varphi,\theta,0)\mathbf{n}_0 = \mathbf{n}$, $\mathbf{n}_0 = (0,0,1)$, where $g(\varphi,\theta,\psi)$ is defined in Equation 11.8. The corresponding symplectic structure is defined by the Poisson brackets

$$\{f,g\} = \frac{1}{\sin\theta}\left(\partial_\varphi f\,\partial_\theta g - \partial_\theta f\,\partial_\varphi g\right)$$

The canonical coordinates are $(\varphi, \cos\theta)$: $\{\varphi, \cos\theta\} = 1$ and the invariant 2-form is $\omega = \sin\theta\, d\theta \wedge d\varphi$.

The elements of the quotient space $su(2)/U(1)$ can be represented in the form $\exp\left[-\theta/2\left(S_+ \mathrm{e}^{-\mathrm{i}\varphi} - S_- \mathrm{e}^{\mathrm{i}\varphi}\right)\right]$. Since any element $T(g)$ in $su(2)$ can be factored as a product

$$T(g) = T(\tilde{g})T(h)$$

where $\tilde{g} \in su(2)/U(1)$ and $h \in U(1)$, coherent states for a given irreducible representation are obtained by application of the operator

$$D(\theta, \varphi) = \exp\left[-\frac{\theta}{2}\left(S_+ \mathrm{e}^{-\mathrm{i}\varphi} - S_- \mathrm{e}^{\mathrm{i}\varphi}\right)\right]$$
$$D^\dagger(\theta, \varphi) = D^{-1}(\theta, \varphi) = D(-\theta, \varphi)$$

on the state $|0, N\rangle$.

One can write the operator $D(\theta, \varphi)$ in the disentangled form:

$$D(\xi) = \mathrm{e}^{\xi S_+}\mathrm{e}^{\eta S_z}\mathrm{e}^{-\xi^* S_-}$$

where

$$\xi = -\tan\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\varphi}, \quad \eta = \ln\left(1 + |\xi|^2\right)$$

Since

$$\mathrm{e}^{\eta S_z}|0, N\rangle = \left(1 + |\xi|^2\right)^{-N/2}|0, N\rangle$$

we obtain

$$|\xi\rangle = D(\xi)|0, N\rangle = \left(1 + |\xi|^2\right)^{-N/2}\exp\left[\xi S_+\right]|0, N\rangle$$

Thus, expanding $\exp(\xi S_+)$ in series, we obtain coherent states as an expansion over the basis states of the irreducible representation:

$$|\xi\rangle = \left(1 + |\xi|^2\right)^{-N/2}\sum_{k=0}^{N}\sqrt{\frac{N!}{(N-k)!k!}}\,\xi^k|k, N\rangle \tag{11.17}$$

Alternatively, one can use the angular parameterization

$$|\xi\rangle = |\theta, \varphi\rangle = \sum_{k=0}^{N}\sqrt{\frac{N!}{k!(N-k)!}}\,\mathrm{e}^{-\mathrm{i}k\varphi}\left(-\sin\frac{\theta}{2}\right)^k\cos^{N-k}\frac{\theta}{2}|k, N\rangle$$

It can easily be seen that the change in the parameters (θ, φ) that determine a point on the sphere S^2 to a complex parameter $\xi = -\mathrm{e}^{-\mathrm{i}\varphi}\tan\theta/2$ is simply the stereographic projection of the point on the unit sphere with coordinates (θ, φ) to the complex plane (tangent to the sphere at the north pole) from the south pole.

The coherent states form an overcomplete set: the resolution of unity for the $su(2)$ reads

$$\frac{N+1}{4\pi}\int d\Omega|\theta,\varphi\rangle\langle\theta,\varphi| = I, \quad d\Omega = \sin\theta\, d\theta d\varphi$$

or in the complex plane

$$\frac{N+1}{\pi}\int d\mu_\xi|\xi\rangle\langle\xi| = I, \quad d\mu_\xi = \frac{d^2\xi}{\left(1+|\xi|^2\right)^2}$$

The states $|\theta,\varphi\rangle$ are not orthogonal and can be shown to satisfy

$$|\langle\theta_1,\varphi_1|\theta_2,\varphi_2\rangle|^2 = \left(\frac{1+\mathbf{n}_1\cdot\mathbf{n}_2}{2}\right)^N$$

The coherent states minimize the fluctuations of the Casimir operator $S^2 = S_x^2 + S_y^2 + S_z^2$:

$$\Delta S^2 = \langle S^2\rangle - \sum_{k=x,y,z}\langle S_k\rangle^2 = \frac{N}{2}$$

where the average is taken over coherent states. Thus, in this sense, the coherent states are the states closest to the classical state. Finally, the state $|\xi\rangle$ are eigenstates of the operator $(\mathbf{n}\cdot\mathbf{S})$:

$$(\mathbf{n}\cdot\mathbf{S})|\xi\rangle = -\frac{N}{2}|\xi\rangle$$

which is a simple consequence of the transformation property $D(\xi)S_zD^\dagger(\xi) = (\mathbf{n}\cdot\mathbf{S})$.

11.3
Linear Systems

We say that a quantum system is *linear* if its Hamiltonian is a linear form on the generators of a representation of some (finite dimensional) Lie algebra:

$$H = \sum_j \mu_j(t,\mathbf{C})\, L_j \tag{11.18}$$

where $\mu_i(t,\mathbf{C})$ are the coefficients that can depend on the time and, also, on the Casimir operators $\mathbf{C}$ of the algebra, and

$$[L_j, L_k] = \sum_l c_{jkl}\, L_l \tag{11.19}$$

Let us denote the corresponding representation of the Lie algebra by $T(g)$.

Applying transformations from the group to the elements of the algebra,

$$T(g)L_jT^\dagger(g) = \sum_k \alpha_{jk}(g)L_k \tag{11.20}$$

we obtain the representation of the Lie group acting in the Lie algebra space. In other words, the the elements of the algebra are transformed among themselves. For instance, if L_j are the $su(2)$ generators, we have

$$T(g)S_jT^{\dagger}(g) = \sum_k S_k D^1_{kj}(g)$$

and "quantum" transformation of operators is reduced to the common rotation of the (operator) vector in the real space. This is the crucial point that allows us to treat analytically the systems described with Hamiltonian of the type (11.18). We consider two cases:

1. We need to find a spectrum of the the time-independent Hamiltonian of type (11.18). In this case, owing to (11.20), a unitary transformation (an operator from the group representation) exists that diagonalizes the Hamiltonian (11.18). In other words, there is an operator $T(g)$ such that

   $$T(g)HT^{-1}(g) = \sum_k \alpha_k h_k = H_{\mathrm{d}}$$

   where h_k are elements of the Cartan subalgebra. From here, it immediately follows that the spectrum of a linear system is equidistant, or is a superposition of equidistant spectra. The eigenstates $|\psi_n\rangle$ of the Hamiltonian (11.18) are connected with the eigenstates $|\widetilde{\psi}_n\rangle$ of the diagonal Hamiltonian H_d in the following way:

   $$|\psi_n\rangle = T^{-1}(g)|\widetilde{\psi}_n\rangle$$

2. The Hamiltonian (11.18) depends explicitly on time (time-independent Hamiltonians obviously are also included as a particular case) and it is required to find the evolution operator $U(t, t_0)$ that satisfies the Schrödinger equation

   $$\mathrm{i}\dot{U}(t, t_0) = HU(t, t_0), \;\; U(t_0, t_0) = I \tag{11.21}$$

   Owing to the property (11.20), the evolution operator can be represented in the form of the product

   $$U(t, t_0) = \Pi_k \exp\left[\lambda_k(t) L_k\right] \tag{11.22}$$

   where the coefficients $\lambda_k(t)$ are solutions of a system of first- order ordinary differential equations (in general, nonlinear) that are obtained by substituting the evolution operator in the form (11.22) into the Schrödinger equation and they are equal to the coefficients of the generators themselves.

We apply these two approaches to the following examples:

1. A harmonic oscillator under the influence of an external (time-dependent) force, which is described by the Hamiltonian

$$H = \omega a^{\dagger} a + f(t) a + f(t)^{*} a^{\dagger} \tag{11.23}$$

2. N two-level atoms prepared in a symmetric state in an external field, described by the Hamiltonian

$$H = \omega S_z + f(t)\, S_{+} + f(t)^{*}\, S_{-}, \tag{11.24}$$

where the operators $S_z,\ S_{+},\ S_{-}$ are generators of the $su(2)$ algebra.

11.3.1
Diagonalization of the Time-independent Hamiltonian

1. Harmonic Oscillator with a Constant Force $f(t)=const$)

The Hamiltonian (11.23) is linear on the elements of the oscillatory algebra. The unitary transformation

$$U = \exp\left[\frac{a^{\dagger} f}{\omega} - \frac{a f}{\omega}\right] = D\left(\frac{f}{\omega}\right)$$

diagonalizes the Hamiltonian:

$$U H U^{\dagger} = \omega\, a^{\dagger} a - \frac{|f|^2}{\omega}$$

Eigenstates $|\psi_n\rangle$ of the Hamiltonian (11.23) are related to the standard basis $|n\rangle$, $a^{\dagger} a |n\rangle = n|n\rangle$ in the following way:

$$|\psi_n\rangle = D(-f/\omega)|n\rangle$$

2. Atomic System in a Constant External Field

To diagonalize the Hamiltonian (11.24), it is convenient to rewrite it in terms of Hermitian operators S_x, S_y, S_z:

$$H = \omega S_z + g_x S_x + g_y\, S_y \tag{11.25}$$

where $g_x = 2\mathrm{Re} f,\ g_y = -2\mathrm{Im} f$. The method consists of two steps. First we apply the transformation

$$U_1 = \exp\left(\mathrm{i}\lambda\, S_x\right)$$

to the Hamiltonian (11.25), where the parameter λ is selected in such a way that the terms proportional to S_y in the transformed Hamiltonian cancel out. Using the transformation rules

$$\mathrm{e}^{\mathrm{i}\lambda\, S_x} S_z \mathrm{e}^{-\mathrm{i}\lambda\, S_x} = S_z \cos\lambda + \; S_y \sin\lambda$$

$$\mathrm{e}^{\mathrm{i}\lambda\, S_x} S_y \mathrm{e}^{-\mathrm{i}\lambda\, S_x} = S_y \cos\lambda - \; S_z \sin\lambda$$

we obtain

$$H_1 = U_1 H U_1^\dagger = \left(\omega \cos\lambda - g_y \sin\lambda\right) S_z + g_x S_x, \quad \tan\lambda = -g_y/\omega$$

Now we apply the second transformation

$$U_2 = \exp\left(\mathrm{i}\mu\, S_y\right)$$

where the parameter μ is selected in such a way that the terms proportional to S_x in the transformed Hamiltonian with U_2 are canceled out. Using

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\mu\, S_y} S_z \mathrm{e}^{-\mathrm{i}\mu\, S_y} &= S_z \cos\mu - \ S_x \sin\mu \\ \mathrm{e}^{\mathrm{i}\mu S_y} S_x \mathrm{e}^{-\mathrm{i}\mu\, S_y} &= S_x \cos\mu + \ S_z \sin\mu \end{aligned}$$

we obtain

$$\begin{aligned} H_2 = U_2\, H_1\, U_2^\dagger &= \left[\left(\omega\cos\lambda - g_y \sin\lambda\right)\cos\mu + g_x \sin\mu\right] S_z \\ \cot\mu &= \frac{g_x}{\sqrt{\omega^2 + g_y^2}} \end{aligned}$$

Finally, the diagonalized Hamiltonian takes the form

$$H_2 = \Omega\, S_z, \ \Omega = \sqrt{\omega^2 + g_x^2 + g_y^2}$$

The eigenstates $|\psi\rangle$ of the Hamiltonian (11.25) are related to the states of the standard basis $|k, N\rangle$ by

$$|\psi\rangle = U_1^\dagger\, U_2^\dagger |k, N\rangle$$

11.3.2
Evolution Operator

1. Harmonic Oscillator with a Time-dependent Force

Let us look for the evolution operator in the disentangled normal form

$$U(t) = C(t)\mathrm{e}^{\alpha(t)\, a^\dagger a}\mathrm{e}^{\beta(t)\, a^\dagger}\, \mathrm{e}^{\gamma(t)\, a} \tag{11.26}$$

Substituting (11.26) into equation (11.21) we obtain

$$\begin{aligned} &\mathrm{i}\left(\frac{\dot C}{C} + \dot\alpha\, a^\dagger a + \dot\beta \mathrm{e}^{\alpha\, a^\dagger a} a^\dagger \mathrm{e}^{-\alpha\, a^\dagger a} + \dot\gamma \mathrm{e}^{\alpha\, a^\dagger a} \mathrm{e}^{\beta a^\dagger} a \mathrm{e}^{-\beta\, a^\dagger}\, \mathrm{e}^{-\alpha\, a^\dagger a}\right) U(t) \\ &\quad = (\omega a^\dagger a + f(t) a + f(t)^* a^\dagger)\, U(t) \end{aligned}$$

Using the relations

$$\mathrm{e}^{\beta a^\dagger} a \mathrm{e}^{-\beta\, a^\dagger} = a - \beta, \quad \mathrm{e}^{\alpha\, a^\dagger a} a^\dagger \mathrm{e}^{-\alpha\, a^\dagger a} = a^\dagger \mathrm{e}^{\alpha}, \quad \mathrm{e}^{\alpha\, a^\dagger a} a \mathrm{e}^{-\alpha\, a^\dagger a} = a \mathrm{e}^{-\alpha}$$

and taking into account that $U(t)$ never takes zero values, we obtain

$$\mathrm{i}\frac{\dot{C}}{C} + \mathrm{i}\dot{\alpha}\, a^{\dagger}a + \mathrm{i}\dot{\beta}\mathrm{e}^{\alpha}a^{\dagger} + \mathrm{i}\dot{\gamma}\left(a\mathrm{e}^{-\alpha} - \beta\right) = \omega a^{\dagger}a + f(t)a + f(t)^{*}a^{\dagger}$$

We now equal the coefficients of the operators themselves. This leads to the following system of ordinary differential equations:

$$\frac{\dot{C}}{C} = \dot{\gamma}\beta, \quad \mathrm{i}\dot{\alpha} = \omega, \quad \mathrm{i}\dot{\beta}\mathrm{e}^{\alpha} = f(t)^{*}, \quad \mathrm{i}\dot{\gamma}\mathrm{e}^{-\alpha} = f(t)$$

The parameters should satisfy the initial conditions $C(0) = 1,\ \alpha(0) = \beta(0) = \gamma\#(0) = 0$ (which are the consequences of the condition $U(0) = I$).

2. Atomic System in an External Field

We look for the evolution operator in the normal disentangled form

$$U(t) = \mathrm{e}^{\alpha(t)S_{+}}\mathrm{e}^{\beta(t)S_{z}}\mathrm{e}^{\gamma(t)S_{-}} \tag{11.27}$$

Substituting Equation 11.27 into Equation 11.21, we obtain

$$\begin{aligned}\mathrm{i}\left(\dot{\alpha}S_{+} + \dot{\beta}\mathrm{e}^{\alpha S_{+}}S_{z}\mathrm{e}^{-\alpha S_{+}} + \dot{\gamma}\mathrm{e}^{\alpha S_{+}}\mathrm{e}^{\beta S_{z}}S_{-}\mathrm{e}^{-\beta S_{z}}\mathrm{e}^{-\alpha S_{+}}\right)U(t)\\ = \left(\omega_{0}S_{z} + f(t)S_{+} + f(t)^{*}S_{-}\right)U(t)\end{aligned}$$

Using the following relations

$$\begin{aligned}\mathrm{e}^{\alpha S_{+}}S_{z}\mathrm{e}^{-\alpha S_{+}} &= S_{z} - \alpha S_{+}\\ \mathrm{e}^{\beta S_{z}}S_{-}\mathrm{e}^{-\beta S_{z}} &= \mathrm{e}^{-\beta}S_{-}\\ \mathrm{e}^{\alpha S_{+}}S_{-}\mathrm{e}^{-\alpha S_{+}} &= S_{-} + 2\alpha S_{z} - \alpha^{2}S_{+}\end{aligned}$$

we obtain

$$\begin{aligned}\mathrm{i}\dot{\alpha}S_{+} + \mathrm{i}\dot{\beta}\left(S_{z} - \alpha S_{+}\right) + \mathrm{i}\dot{\gamma}\mathrm{e}^{-\beta}\left(S_{-} + 2\alpha S_{z} - \alpha^{2}S_{+}\right)\\ = \omega_{0}S_{z} + f(t)S_{+} + f(t)^{*}S_{-}\end{aligned}$$

Now we equal the coefficients of the same operators. This leads us to the system of ordinary differential equations

$$\mathrm{i}\left(\dot{\alpha} - \dot{\beta}\alpha - \dot{\gamma}\mathrm{e}^{-\beta}\alpha^{2}\right) = f(t), \quad \mathrm{i}\left(\dot{\beta} + 2\dot{\gamma}\mathrm{e}^{-\beta}\alpha\right) = \omega_{0}, \quad \mathrm{i}\dot{\gamma}\mathrm{e}^{-\beta} = f(t)^{*}$$

which should satisfy the initial conditions $\alpha(0) = \beta(0) = \gamma(0) = 0$ (these are the consequences of the condition $U(0) = 1$).

11.3.3
Reference Formulas

For reference purposes, we list the decomposition formulae (which can be obtained by using the methods of this appendix) of the generic element for different groups.

H(1) *group:*

$$\exp\left[\alpha_+ a^\dagger + \alpha_- a\right] = \exp\left(\frac{-|\alpha|^2}{2}\right)\exp\left(\alpha_+ a^\dagger\right)\exp\left(\alpha_- a\right)$$

su(2) *group:*

$$\exp\left[\omega S_z + \alpha_+ S_+ + \alpha_- S_-\right] = \exp\left(\beta_+ S_+\right)\exp\left(\ln\beta_0 S_z\right)\exp\left(\beta_- S_-\right) \tag{11.28}$$

where

$$\beta_0 = \left(\cosh\Omega - \frac{\omega}{2\Omega}\sinh\Omega\right)^{-2}, \quad \beta_\pm = \frac{2\alpha_\pm \sinh\Omega}{2\Omega\cosh\Omega - \omega\sinh\Omega},$$

$$\Omega^2 = \frac{\omega^2}{4} + \alpha_+\alpha_-$$

su(1, 1) *group:*

$$\exp\left[\omega K_0 + \alpha_+ K_+ + \alpha_- K_-\right] = \exp\left(\gamma_+ K_+\right)\exp\left(\ln\gamma_0 K_0\right)\exp\left(\gamma_- K_-\right) \tag{11.29}$$

where

$$\gamma_0 = \left(\cosh\Xi - \frac{\omega}{2\Xi}\sinh\Xi\right)^{-2}, \quad \gamma_\pm = \frac{2\alpha_\pm \sinh\Xi}{2\Xi\cosh\Xi - \omega\sinh\Xi},$$

$$\Xi^2 = \frac{\omega^2}{4} - \alpha_+\alpha_-$$

Integration of a Gaussian function over the complex plane:

$$\frac{1}{\pi}\int d^2z\,\exp\left(\alpha z + \beta z^* - \gamma|z|^2\right) = \frac{1}{\gamma}\exp\left(\frac{\alpha\beta}{\gamma}\right)$$

11.4
Lie Transformation Method

In this appendix, we briefly describe the Lie transformation method (which we will also call ''small rotations''). The small-rotation method [24], [124] offers a *regular* procedure to approximately obtain Hamiltonians that describe effective dynamics of nonlinear quantum systems. The idea of this method is based on the observation that various Hamiltonians in quantum optics can be written in terms of polynomially deformed algebras $sl_{\mathrm{pd}}(2, R)$ [92]

$$H_{\mathrm{int}} = \Delta\, X_0 + g\left(X_+ + X_-\right) \tag{11.30}$$

where the operators $X_\pm$ and X_0 are generators of the deformed algebra and satisfy the following commutation relations:

$$[X_0, X_\pm] = \pm X_\pm, \quad [X_+, X_-] = P(X_0) \tag{11.31}$$

where $P(X_0)$ is a polynomial function of the diagonal operator X_0 with coefficients that depend on some integrals of motion N_j. If $P(X_0)$ is a linear function of X_0, the usual $sl(2, R)$ or $su(2)$ algebras are obtained. If for some physical reason (which depends on the particular model being considered), $\eta = g/\Delta \ll 1$ is a small parameter, the Hamiltonian (11.30) is *almost* diagonal in the basis of the eigenstates of X_0 and can be approximately diagonalized by applying the following unitary transformation in a perturbative way (a *small nonlinear rotation*)

$$U = \exp\left[\eta\left(X_+ - X_-\right)\right] \tag{11.32}$$

By applying the transformation (11.32) to the Hamiltonian (11.30), and according to the standard expansion

$$e^{\eta A} B e^{-\eta A} = \sum_{k=1}^{\infty} \frac{\eta^k}{k!} \mathrm{ad}_A^k(B) \tag{11.33}$$

where ad_A is the adjoint operator defined as $\mathrm{ad}_A(B) = [A, B]$, we obtain

$$H_{\mathrm{eff}} = U H_{\mathrm{int}} U^{\dagger} = \Delta\, X_0 + g \sum_{k=1}^{\infty} \eta^k \frac{k}{(k+1)!} \mathrm{ad}_T^k(X_+ + X_-) \tag{11.34}$$

where $T = X_+ - X_-$. Considering that, owing to Equation 11.31

$$\mathrm{ad}_T(X_0) = [T, X_0] = -\left(X_+ + X_-\right)$$

the effective Hamiltonian acquires the following form:

$$H_{\mathrm{eff}} = \Delta\, X_0 + g\eta \sum_{k=0}^{\infty}{}' \eta^k \left[X_+^k f_k(X_0, \eta) + \mathrm{h.c.}\right] \tag{11.35}$$

where $f_k(X_0, \eta)$ is a function of the diagonal operator X_0 and can be represented as a series on η:

$$f_k(X_0, \eta) = \frac{2\,(k+1)}{(k+2)!} \nabla^{k+1} \phi(X_0) + \mathcal{O}(\eta)$$

where $\phi(X_0) = X_+ X_-$ is the so-called structural function, so that $\nabla\phi(X_0) = \phi(X_0) - \phi(X_0 + 1) = P(X_0)$; ($'$) in Equation 11.35 means that the term with $k = 0$ is taken with the coefficient 1/2. The unitary character of (11.32) ensures that H_{eff} has the same spectrum as H_{int} (so they are physically equivalent).

Conserving only terms up to the order η, we obtain

$$H_{\mathrm{eff}} = \Delta\, X_0 + \eta g \nabla\phi(X_0) \tag{11.36}$$

and, in a first approximation, the resulting effective Hamiltonian is diagonal in the basis of the eigenstates of X_0.

The higher order contributions always have the form $X_+^k f_k(X_0) + \mathrm{h.c.} + g(X_0)$. This makes the procedure of eliminating nondiagonal terms at each step trivial, in the sense that it is always obvious that a transformation should be applied. For example, to eliminate terms of the form

$$\eta^l[X_+^k f_k(X_0) + f_k(X_0)X_-^k]$$

it is sufficient to apply the transformation

$$\exp\left(\frac{\eta^{l+1}}{k}T_k\right) \tag{11.37}$$

with $T_k = X_+^k f_k(X_0) - f_k(X_0)X_-^k$, since the first commuter of $\Delta\ X_0$ with $(\eta^{l+1}/k)T_k$ cancels out the corresponding term in the Hamiltonian.

Since the transformed Hamiltonian is of the form of an expansion of the small parameter η, it is clear that its eigenvalues coincide with those obtained using the standard perturbative theory. The advantage of this method consists in obtaining *effective Hamiltonians* in an operator form, which allows us to avoid the time-consuming calculation of series of matrix elements.

This technique also allows us to obtain corrections to the eigenvalues of the Hamiltonian (11.30) in a simple way. Indeed, it can be easily seen that the eigenvalues of the interaction Hamiltonian (11.30) are approximated as

$$|\Psi_m\rangle = U^\dagger|m\rangle$$

where $|m\rangle$ is an eigenvalue of the diagonal operator X_0 and U is the corresponding small rotation operator. Since the rotation operators and $|m\rangle$ are time independent, the operator U can be applied to $|m\rangle$ in the form of an expansion of η. For example, the eigenstate $|\Psi_m\rangle$ up to order η^2 is of the form

$$|\Psi_m\rangle = |m\rangle - \eta(X_+ - X_-)|m\rangle + \frac{\eta^2}{2}\{(X_+^2 + X_-^2) - [\phi(X_0) + \phi(X_0+1)]\}|m\rangle$$

This representation is specially advantageous if we construct the state space for the model as a space representation of the deformed *su*(2) algebra using the raising operator X_+: $|m\rangle \propto X_+^m|0\rangle$, where $|0\rangle$ is the lowest vector that satisfies the standard condition $X_-|0\rangle = 0$.

11.5
Wigner *d* Function

In this appendix, we provide an overview of some properties of the Wigner *d* function. In the Euler parameterization, the finite rotation element of the $SU(2)$ group is written as

$$g(\varphi,\theta,\psi) = \mathrm{e}^{-\mathrm{i}\varphi S_z}\mathrm{e}^{-\mathrm{i}\theta S_y}\mathrm{e}^{-\mathrm{i}\psi S_z}$$

and its matrix elements in the standard basis of the angular momentum $|p,S\rangle$, $k,p = -S,\ldots,S$ have the form

$$D_{kp}^S(\varphi,\theta,\psi) = \langle k,S|T(g)|p,S\rangle = \mathrm{e}^{-\mathrm{i}(\varphi k+\psi p)}\langle k,S|\mathrm{e}^{-\mathrm{i}\theta S_y}|p,S\rangle \tag{11.38}$$

Here, matrix elements that correspond to the rotation around the y axis are real; they are called *Wigner $d^S(\theta)$ functions*

$$d^S_{kp}(\theta) = \langle k, S|e^{-i\theta S_y}|p, S\rangle \tag{11.39}$$

$$= \sqrt{\frac{(S-k)!(S-p)!}{(S+k)!(S+p)!}} \cot^{k+p}\theta/2 \sum_{j=\max(k,p)}^{S} \frac{(-1)^{j-k}(S+j)!\sin^{2j}\theta/2}{(S-j)!(j-k)!(j-p)!}$$

Functions $D^S_{kp}(\varphi, \theta, \psi)$ satisfy the following orthogonality relations:

$$\int dV D^S_{kp}(\varphi, \theta, \psi)^* D^{S'}_{k'p'}(\varphi, \theta, \psi) = \frac{16\pi^2}{2S+1}\delta_{kk'}\delta_{pp'}\delta_{SS'}$$

where the volume element of the $SU(2)$ group is $dV = \sin\theta\, d\theta\, d\varphi\, d\psi$, and the completeness relations

$$\sum_{S,k,p}(2S+1)\, D^S_{kp}(\varphi, \theta, \psi)^* D^S_{kp}(\varphi', \theta', \psi')$$

$$= 16\pi^2\delta\left(\varphi-\varphi'\right)\delta\left(\cos\theta - \cos\theta'\right)\delta\left(\psi - \psi'\right)$$

Therefore, any function on the group $SU(2)$ can be expanded in the basis $D^S_{kp}(\varphi, \theta, \psi)$.

The element $D^S_{kp}(\varphi, \theta, \psi)$ is related to the spherical harmonics in the following way,

$$D^S_{k0}(\varphi, \theta, \psi)^* = \sqrt{\frac{4\pi}{2S+1}}Y_{Sk}(\varphi, \theta), \quad S = 0, 1, 2, \ldots \quad -S \le k \le S \tag{11.40}$$

which are essentially the associated Legendre polynomials $P^k_S(\cos\theta)$,

$$Y_{Sk}(\varphi, \theta) = (-1)^k\sqrt{\frac{(2S+1)\left(S-k\right)!}{4\pi\left(S+k\right)!}}P^k_S(\cos\theta)e^{ik\varphi} \tag{11.41}$$

$$Y^*_{Sk}(\varphi, \theta) = (-1)^k Y_{S-k}(\varphi, \theta)$$

In particular,

$$Y_{S0}(\varphi, \theta) = \sqrt{\frac{2S+1}{4\pi}}\; P_S(\cos\theta)$$

where $P_S(\cos\theta)$ are the Legendre polynomials. The spherical harmonics form an orthonormal basis for the measure $d\Omega = \sin\theta\; d\theta\; d\varphi$ on the sphere S^2

$$\int Y_{Sk}(\varphi, \theta)Y^*_{S'm}(\varphi, \theta)d\Omega = \delta_{km}\delta_{SS'} \tag{11.42}$$

and, in particular, when $S' = 0$

$$\int Y_{Sk}(\varphi, \theta)d\Omega = \sqrt{4\pi}\,\delta_{k0}\delta_{S0} \tag{11.43}$$

For a given group element g, the angles (φ, θ) define a point on the sphere with coordinates $\mathbf{n} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$, so that the spherical harmonics can be written as $Y_{Sk}(\vec{n})$. From the group multiplication low

$$g(\varphi, \theta, \psi)g(\varphi_0, \theta_0, \psi_0) = g(\varphi_1, \theta_1, \psi_1) \tag{11.44}$$

it follows that the unit vector $\mathbf{n}$ is transformed as $g \cdot \mathbf{n}_0 = \mathbf{n}_1$, where explicit formulas for the components of $\mathbf{n}_0$ can easily be found. Rewriting Equation 11.44 in the group representation, we have

$$\sum_m D^S_{pm}(\varphi, \theta, \psi) D^S_{m0}(\varphi_0, \theta_0, \psi_0) = D^S_{p0}(\varphi_1, \theta_1, \psi_1)$$

From Equation (11.40, we find that the spherical harmonics are transformed as follows:

$$Y_{Sp}(g \cdot \mathbf{n}_0) = \sum_{m=-S}^{S} Y_{Sm}(\mathbf{n}_0) D^S_{mp}(g) \tag{11.45}$$

In applications, the following summation rule is important:

$$\sum_{k=-S}^{S} Y_{Sk}(\varphi, \theta) Y^*_{Sk}(\varphi', \theta') = \frac{4\pi}{2S+1} P_S(\cos\omega)$$

where ω is the angle between the points $\mathbf{n} = \mathbf{n}(\varphi, \theta)$ and $\mathbf{n}' = \mathbf{n}(\varphi', \theta')$, $\cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$.

The analog of the δ function in the sphere (the reproductive kernel) is the sum

$$\sum_{l=0}^{2S} \sum_{k=-l}^{l} Y_{lk}(\mathbf{n}) Y^*_{lk}(\mathbf{n}') = \delta(\mathbf{n}, \mathbf{n}'), \quad \mathbf{n}, \mathbf{n}' \in S^2 \tag{11.46}$$

such that

$$\int f(\mathbf{n})\delta(\mathbf{n}, \mathbf{n}')d\Omega = f(\mathbf{n}')$$

for any function $f(\mathbf{n})$ defined on the sphere.

In terms of the Dicke states, which form a basis in $(N+1)$-dimensional Hilbert space of the atomic system, the d function

$$d^N_{mn}(\theta) = \langle m, N|\, e^{-i\theta S_y}\, |n, N\rangle$$

is obtained from the formula (11.39) by substituting $N = 2S$, $k = m - N/2$, $p = n - N/2$, and changing the summation index ($j \to l = N/2 - j$):

$$d^N_{mn}(\theta) = \mathcal{N}_{nm} \cot^{m+n-N} \frac{\theta}{2} \sum_{l=0}^{\min(N-m,N-n)} \frac{(-1)^{N-n-l}(N-l)!\sin^{N-2l}\theta/2}{l!\,(N-l-m)!\,(N-l-n)!}$$

where

$$\mathcal{N}_{nm} = \sqrt{\frac{(N-m)!\,(N-n)!}{m!n!}}$$

For $N = 1$,

$$\mathrm{d}^1(\theta) = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

For $N = 2$,

$$S_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{bmatrix},$$

$$\mathrm{d}^2(\theta) = \frac{1}{2} \begin{bmatrix} 1+\cos\theta & -\sqrt{2}\sin\theta & 1-\cos\theta \\ \sqrt{2}\sin\theta & 2\cos\theta & -\sqrt{2}\sin\theta \\ 1-\cos\theta & \sqrt{2}\sin\theta & 1+\cos\theta \end{bmatrix}$$

Sometimes, instead of using matrix elements $\langle m, N|\, \mathrm{e}^{-\mathrm{i}\theta S_y}\, |n, N\rangle$, which are real but not symmetric, the symmetric matrix elements $\left\langle m, N|\mathrm{e}^{\mathrm{i}\theta S_x}|n, N\right\rangle$ called the $P^N_{mn}(\theta)$ function are used

$$P^N_{mn}(\theta) = \left\langle m, N|\mathrm{e}^{\mathrm{i}\theta S_x}|n, N\right\rangle$$

It is easy to find the relation between the d and P functions. Owing to the relation

$$S_x = \mathrm{e}^{\mathrm{i}S_z\pi/2}\, S_y\, \mathrm{e}^{-\mathrm{i}S_z\pi/2}$$

we have

$$P^N_{mn}(\theta) = \mathrm{i}^{m-n} d^N_{mn}(-\theta)$$

For example,

$$P^1(\theta) = \begin{bmatrix} \cos(\theta/2) & \mathrm{i}\sin(\theta/2) \\ \mathrm{i}\sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

$$P^2(\theta) = \begin{bmatrix} \dfrac{1+\cos\theta}{2} & \mathrm{i}\dfrac{\sin\theta}{\sqrt{2}} & -\dfrac{1-\cos\theta}{2} \\ \mathrm{i}\dfrac{\sin\theta}{\sqrt{2}} & \cos\theta & \mathrm{i}\dfrac{\sin\theta}{\sqrt{2}} \\ -\dfrac{1-\cos\theta}{2} & \mathrm{i}\dfrac{\sin\theta}{\sqrt{2}} & \dfrac{1+\cos\theta}{2} \end{bmatrix}$$

The P function is symmetric: $P^N_{mn}(\theta) = P^N_{nm}(\theta)$. It is also clear that

$$P^N_{NN}(\theta) = \cos^N \tfrac{\theta}{2}, \quad P^N_{N0}(\theta) = \mathrm{i}^N \sin^N \tfrac{\theta}{2}$$

$$P^N_{Nk}(\theta) = \mathrm{i}^{N-k} \sqrt{\tfrac{N!}{(N-k)!k!}}\, \sin^{N-k} \tfrac{\theta}{2}\, \cos^k \tfrac{\theta}{2}$$

The summation rule for the P functions has the form

$$\sum_{k=0}^{N} P_{mk}^{N}(\theta_1)P_{kn}^{N}(\theta_2)\,\mathrm{e}^{-\mathrm{i}k\xi} = \mathrm{e}^{-\mathrm{i}\xi N/2}\mathrm{e}^{\mathrm{i}(\varphi+\psi)N/2}\mathrm{e}^{-\mathrm{i}(\varphi m+\psi n)}\,P_{mn}^{N}(\theta)$$

where

$$\begin{aligned}
\cos\theta &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos\xi\\
\exp(\mathrm{i}\varphi) &= \frac{\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\cos\xi + \mathrm{i}\sin\theta_2\sin\xi}{\sin\theta}\\
\exp\left[\mathrm{i}\frac{(\varphi+\psi)}{2}\right] &= \frac{\cos\theta_1/2\cos\theta_2/2\,\mathrm{e}^{\mathrm{i}\xi/2} - \sin\theta_1/2\sin\theta_2/2\,\mathrm{e}^{-\mathrm{i}\xi/2}}{\cos\theta/2}
\end{aligned}$$

In particular, in the case when $\xi = 0$ and $\theta_1+\theta_2 < \pi$, then $\theta = \theta_1+\theta_2,\ \varphi = \psi = 0$, and the summation formula is transformed into

$$\sum_{k=0}^{N} P_{mk}^{N}(\theta_1)P_{kn}^{N}(\theta_2) = P_{mn}^{N}(\theta_1+\theta_2)$$

If $\xi = 0$ and $\theta_1+\theta_2 > \pi$, then $\theta = 2\pi-\theta_1-\theta_2$ and $\varphi = \psi = \pi$, which yields

$$(-1)^{n+m-N}\sum_{k=0}^{N} P_{mk}^{N}(\theta_1)P_{kn}^{N}(\theta_2) = P_{mn}^{N}(\theta_1+\theta_2)$$

It can easily be seen that the following relation holds:

$$\sum_{k=0}^{N} P_{mk}^{N}(\theta)P_{kn}^{N}(-\theta) = P_{mn}^{N}(0) = \delta_{mn}$$

The following integral representation for the function $P_{mk}^{N}(\theta)$ is useful in applications:

$$\begin{aligned}
P_{mk}^{N}(\theta) &= \frac{1}{2\pi}\sqrt{\frac{(N-k)!\,k!}{(N-m)!\,m!}}\int_0^{2\pi}\mathrm{d}z\,\mathrm{e}^{\mathrm{i}(k-N/2)z}\\
&\times\left[\cos\frac{\theta}{2}\mathrm{e}^{\mathrm{i}z/2}+\mathrm{i}\sin\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}z/2}\right]^{N-m}\left[\cos\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}z/2}+\mathrm{i}\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}z/2}\right]^{m}
\end{aligned}$$

The related sum rules are

$$\begin{aligned}
\sum_{m=0}^{N}\frac{d_{mn}^{N}(\theta)\,z^m}{\sqrt{m!\,(N-m)!}} &= \frac{1}{\sqrt{n!\,(N-n)!}}\left(z\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^{n}\\
&\times\left(z\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^{N-n}
\end{aligned}$$

$$\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{N!\,d_{mn}^{N}(\theta)\,z_1^m z_2^n}{\sqrt{m!n!\,(N-m)!\,(N-n)!}}$$
$$=\left[z_1\sin\frac{\theta}{2}+\cos\frac{\theta}{2}+z_2\left(z_1\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\right]^N$$

11.6
Irreducible Tensor Operators

In this appendix, we use the angular momentum basis $|p,S\rangle$, $k,p=-S,\ldots,S$.

The irreducible tensor operators (ITO) for the $su(2)$ group are defined as follows:

$$\hat{T}_{LM}^{(S)}=\sqrt{\frac{2L+1}{2S+1}}\sum_{m,m'=-S}^{S}C_{S,m;\,L,M}^{S,m'}\,|S,m'\rangle\ \langle S,m| \tag{11.47}$$

where $C_{S,m;\,L,M}^{S,m'}$ is the Clebsch–Gordan coupling coefficient for two angular momenta of size S giving the total momentum $0\le L\le 2S$ with the projection$-L\le M\le L$. The ITO satisfy the following resolution of the identity:

$$\sum_{L=0}^{2S}\sum_{M=-L}^{L}\hat{T}_{L,M}^{(S)\dagger}\hat{T}_{L,M}^{(S)}=I \tag{11.48}$$

Under the transformations of the group, the ITO are transformed in the following way:

$$T(g)\hat{T}_{LM}^{(S)}T^{-1}(g)=\sum_{K=-L}^{L}D_{KM}^{(L)}(g)\hat{T}_{LK}^{(S)} \tag{11.49}$$

where $D_{KM}^{(L)}(g)$ are the matrix elements of $T(g)$ (Equation 11.38).

The ITO satisfy the following commutation relation with the generators of the $su(2)$ algebra:

$$\left[S_z,\hat{T}_{LM}^{(S)}\right]=M\hat{T}_{LM}^{(S)},$$
$$\left[S_\pm,\hat{T}_{LM}^{(S)}\right]=\sqrt{(L\pm M+1)\,(L\mp M)}\hat{T}_{LM\pm1}^{(S)}$$

The ITO form the basis in the space of operators acting in $(2S+1)$- dimensional Hilbert space, so that any $(2S+1)\times(2S+1)$ matrix can be represented as a linear combination of the ITO:

$$\hat{A}=\sum_{L=0}^{2S}\sum_{M=-L}^{L}a_{LM}\hat{T}_{L,M}^{(S)}$$

where

$$a_{LM} = \mathrm{Tr}\left(\hat{A}\,\hat{T}^{(S)\dagger}_{LM}\right), \quad \hat{T}^{(S)\dagger}_{L,M} = (-1)^M\,\hat{T}^{(S)}_{L-M}$$

If the operator $\hat{A}$ is self-conjugated, $\hat{A} = \hat{A}^\dagger$, then $a^*_{LM} = (-1)^M\,a_{L-M}$. In particular, we have

$$S_x = A_S\left(\hat{T}_{1-1} - \hat{T}_{11}\right), \quad S_y = \mathrm{i}A_S\left(\hat{T}_{1-1} + \hat{T}_{11}\right) \tag{11.50}$$

$$S_z = \sqrt{2}A_S\hat{T}_{10}, \quad A_S = \sqrt{\frac{S\,(S+1)\,(2S+1)}{6}}$$

$$I = \sqrt{2S+1}\hat{T}_{00} \tag{11.51}$$

An arbitrary element of the $(2S+1)$-dimensional irreducible representation of the $su(2)$ group is expanded in the basis of the ITO as follows:

$$\mathrm{e}^{-\mathrm{i}y\cdot\mathbf{S}} = \frac{\sqrt{4\pi}}{\sqrt{2S+1}}\sum_{L=0}^{2S}\sum_{M=-L}^{L}(-\mathrm{i})^L\,\chi^S_L(\omega)\,Y^*_{L,M}(\mathbf{n})\,\hat{T}^{(S)}_{L,M} \tag{11.52}$$

where $y = \omega\mathbf{n}$, $\mathbf{n} = (\cos\varphi\sin\theta,\ \sin\varphi\sin\theta, \cos\theta)$, $\mathbf{S} = (S_x, S_y, S_z)$ and $\chi^S_L(\omega)$ are the generalized characters, defined as

$$\chi^S_L(\omega) = \mathrm{i}^L\sum_{M=-S}^{S}\mathrm{e}^{-\mathrm{i}M\omega}C^{SM}_{SM,L0} \tag{11.53}$$

It can be easily seen that $\chi^S_{L=0}(\omega) = \chi^S(\omega)$ is the standard character of the group $su(2)$

$$\chi^S(\omega) = \sum_{M=-S}^{S}\mathrm{e}^{-\mathrm{i}M\omega} = \frac{\sin\left(S+1/2\right)\omega}{\sin\omega/2}$$

The generalized characters $\chi^S_L(\omega)$ have two important orthogonality relations

$$\int_0^{2\pi}\mathrm{d}\omega\,\chi^S_L(\omega)\,\chi^S_{L'}(\omega) = 2\pi\,\delta_{L,L'}\frac{2S+1}{2L+1} \tag{11.54}$$

$$\int_0^{2\pi}\mathrm{d}\omega\sin^2\frac{\omega}{2}\,\chi^S_L(\omega)\,\chi^{S'}_L(\omega) = 2\pi\,\delta_{S,S'} \tag{11.55}$$

We list some other properties of the ITO:

1. The expansion of the product of two ITO

$$\hat{T}^{(S)}_{L_1M_1}\,\hat{T}^{(S)}_{L_2M_2} = \sum_{L=0}^{2S}(-1)^{2S+L}\sqrt{(2L_1+1)\,(2L_2+1)}$$
$$\times\left\{\begin{matrix} L_1 & L_2 & L \\ S & S & S\end{matrix}\right\}C^{LM}_{L_1M_1;L_2M_2}\hat{T}^{(S)}_{LM}$$

2. Traces of ITO products

$$\begin{aligned}
Tr\left(\hat{T}^{(S)}_{LM}\right) &= \sqrt{2S+1}\delta_{L0}\delta_{M0},\\
Tr\left(\hat{T}^{(S)}_{L_1M_1}\hat{T}^{(S)}_{L_2M_2}\right) &= (-1)^{M_1}\delta_{L_1L_2}\delta_{M_1-M_2}\\
Tr\left(\hat{T}^{(S)}_{L_1M_1}\hat{T}^{(S)}_{L_2M_2}\hat{T}^{(S)}_{L_3M_3}\right) &= (-1)^{2S+L_3+M_3}\sqrt{(2L_1+1)(2L_2+1)}\\
&\quad\times\left\{\begin{array}{ccc} L_1 & L_2 & L_3\\ S & S & S\end{array}\right\} C^{L_3-M_3}_{L_1M_1;L_2M_2}\hat{T}^{(S)}_{LM}
\end{aligned}$$

where $\left\{\begin{array}{ccc} L_1 & L_2 & L_3\\ S & S & S\end{array}\right\}$ are the 6-*j* symbols;

3. The “scalar” product of two ITO yields the so-called swapping operator

$$\sum_{L=0}^{2S}\sum_{M=-L}^{L}\left(\hat{T}^{(S)}_{L,M}\right)_{mm'}\left(\hat{T}^{(S)\dagger}_{L,M}\right)_{kk'} = \delta_{mk'}\delta_{m'k} \tag{11.56}$$

The matrix elements of the ITO $\langle\vartheta,\varphi|\,\hat{T}^{(S)}_{L,M}|\vartheta,\varphi\rangle$ in the basis of angular momentum coherent states (10.25),

$$|\vartheta,\varphi\rangle = D^{(S)}(\varphi,\vartheta,0)|S,S\rangle$$

where

$$D^{(S)}(\varphi,\vartheta,0) = \exp(-\mathrm{i}\varphi S_z)\exp(-\mathrm{i}\vartheta S_y)$$

are easily evaluated:

$$\begin{aligned}
\langle\vartheta,\varphi|\,\hat{T}^{(S)}_{L,M}|\vartheta,\varphi\rangle &= \langle S,S|D^{(S)\dagger}(\varphi,\vartheta,0)\,\hat{T}^{(S)}_{L,M}D^{(S)}(\varphi,\vartheta,0)|S,S\rangle\\
&= \sum_{p=-L}^{L} D^{(L)}_{Mp}(\varphi,\vartheta,0)^*\langle S,S|\hat{T}^{(S)}_{L,p}|S,S\rangle\\
&= \sqrt{\frac{2L+1}{2S+1}}\sum_{p=-L}^{L} D^{(L)}_{Mp}(\varphi,\vartheta,0)^* C^{S,S}_{S,S;L,p}\delta_{p0}
\end{aligned}$$

which finally yields

$$\langle\vartheta,\varphi|\,\hat{T}^{(S)}_{L,M}|\vartheta,\varphi\rangle = \sqrt{\frac{4\pi}{2S+1}}C^{S,S}_{S,S;L,0}Y_{LM}(\vartheta,\varphi) \tag{11.57}$$

where $Y_{LM}(\varphi,\vartheta)$ are the spherical harmonics (Equation 11.41) and the coefficient $C^{S,S}_{S,S;L,0}$ is expressed in the closed form as

$$C^{S,S}_{S,S;\,L,0} = (2S)!\left[\frac{2S+1}{(2S-L)!(2S+L+1)!}\right]^{1/2}$$

Equation 11.57 implies that the ITO can also be represented in the form of a decomposition on the coherent states projectors:

$$\hat{T}^{(S)}_{L,M} = \sqrt{\frac{2S+1}{4\pi}}\left(C^{S,S}_{S,S;\,L,0}\right)^{-1}\int d\Omega|\vartheta,\varphi\rangle\langle\vartheta,\varphi|Y_{LM}(\vartheta,\varphi)$$

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